

# Confluence via strong normalisation in an algebraic $\lambda$ -calculus with rewriting

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$M, N ::= x \mid \lambda x.M \mid (M)N \mid M + N \mid \alpha.M \mid 0$

Beta reduction:

$(\lambda x.M)N \rightarrow M[x := N]$

“Algebraic” reductions:

$\alpha.M + \beta.M \rightarrow (\alpha + \beta).M,$

$(M)(N_1 + N_2) \rightarrow (M)N_1 + (M)N_2,$

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*(oriented version of the axioms of vectorial spaces)*

Two origins:

- ▶ Differential  $\lambda$ -calculus: capturing linearity *à la* Linear Logic  
→ *Removing the differential operator*: Algebraic  $\lambda$ -calculus ( $\lambda_{\text{alg}}$ ) [Vaux'09]
- ▶ Quantum computing: superposition of programs  
→ *Linearity as in algebra*: Linear-algebraic  $\lambda$ -calculus ( $\lambda_{\text{lin}}$ )

[Arrighi, Dowek'08]

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	$\lambda_{\text{alg}}$	$\lambda_{\text{lin}}$
<b>Origin</b>	Linear Logic	Quantum computing
<b>Strategy</b>	Call-by-name	Call-by-value

## Confluence issues

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**Solution 3:**

forbid  $\infty$ ! (type system)

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# Possible type systems

## Straightforward extension of a classic type system

$$\frac{\Gamma \vdash M : T \quad \Gamma \vdash N : T}{\Gamma \vdash M + N : T} +_I$$

**Pros:**

- ▶ Simple

$$\frac{\Gamma \vdash M : T}{\Gamma \vdash \alpha.M : T} \alpha_I \quad \frac{}{\Gamma \vdash \mathbf{0} : T} 0_I$$

**Cons:**

- ▶ Too restrictive

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**Scalar type system** [Arrighi, Díaz-Caro 2009]

$$\frac{\Gamma \vdash M : \alpha.T \quad \Gamma \vdash N : \beta.T}{\Gamma \vdash M + N : (\alpha + \beta).T} +_I$$

**Pros:**

- ▶ Characterises the 'amount' of terms

$$\frac{\Gamma \vdash M : T}{\Gamma \vdash \alpha.M : \alpha.T} \alpha_I \quad \frac{}{\Gamma \vdash \mathbf{0} : 0.T} 0_I$$

**Cons:**

- ▶ Still too restrictive

# Possible type systems

**Additive type system** [Díaz-Caro, Petit 2010]

$$\frac{\Gamma \vdash M : T \quad \Gamma \vdash N : R}{\Gamma \vdash M + N : T + R} +_I$$

**Pros:**

- ▶ More versatile
- ▶ Interpretation in System F with pairs

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- ▶ **Overkill** (confluence!)

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**Idea: extend Additive**

Idea:  $\lambda^{\text{CA}}$  - Extend additive

**Key idea**

$$\frac{\Gamma \vdash M : T}{\Gamma \vdash \alpha.M : [\alpha].\bar{T}} = \underbrace{T + \dots + T}_{[\alpha]}$$



# Idea: $\lambda^{\text{CA}}$ - Extend additive

## Key idea

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If  $\vdash M : T$ , then  $\vdash (0.9).M + (1.1).M : T$

$(0.9).M + (1.1).M \rightarrow 2.M$  and  $\vdash 2.M : 2.T$

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## Theorem (weak subject reduction)

$M \rightarrow N, \Gamma \vdash M : T \Rightarrow \Gamma \vdash N : R$  with  $T \preceq R$

# Unicity of types

Second problem:

$$M + M \rightarrow 2.M$$

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Solution: Church style

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- ▶ Strong normalisation:

## Plan

- ▶ Translation from  $\lambda^{\text{CA}}$  to  $\lambda_{\text{lin}}$  (i.e. remove annotations)
- ▶ Preservation of reduction by the translation
- ▶ Typability in  $\lambda^{\text{CA}} \Rightarrow$  Typability in *Vectorial*
- ▶ SN in *Vectorial*  $\Rightarrow$  SN in  $\lambda^{\text{CA}}$

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## Lemma

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$$\Gamma \vdash M : T \Rightarrow \Delta \vdash_v |M| : R$$

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## Proof.

Define:

$$\begin{array}{lll} \sigma(x : U) = 1 & \sigma(\Lambda X.t) = 1 + \sigma(t) & \sigma(t@U) = \sigma(t) \\ \sigma(\lambda x : U.t) = \sigma(t) & \sigma((t) r) = \sigma(t) \sigma(r) & \sigma(\mathbf{0}) = 1 \\ \sigma(\alpha.t) = \sigma(t) & & \sigma(t + r) = \sigma(t) + \sigma(r) \end{array}$$

Induction on  $M$ :  $\sigma((\Lambda X.M)@T) > \sigma(M[X/T])$ . □

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Assume  $M$  not SN:

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Then  $M_i$  must be SN. Absurd. □

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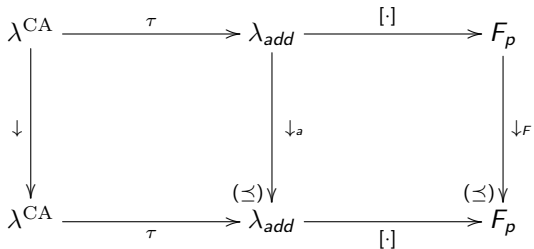
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Lemma (Typing preservation)

$$\Gamma \vdash M : T \Rightarrow \Gamma \vdash_a \tau(M) : T$$

# Abstract interpretation



# Contributions

- ▶ “Powerful” alternative to *Vectorial* (extension of *Additive*)
- ▶ (weak) Subject reduction
- ▶ Strong normalisation (via translation to *Vectorial*)
- ▶ Confluence (via SN)
- ▶ Abstract interpretation in System F with pairs (via *Additive*)



## Possible extensions

- ▶ Taking also the ceil to have intervals...

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- ▶ Complexes...

[Simon Perdrix]

$$\vdash (\alpha - \beta i).M : [(\lfloor \alpha \rfloor, \star), (\star, \lceil \beta \rceil)].T$$

- ▶ etc.