Exercise sheet 4: Primitive recursive functions

Primitive recursive functions

1. Show that for all $n \in \mathbb{N}_0$, the constant function $g_n(x) = n$ is primitive recursive.
2. Show that each of the following are primitive recursive functions.
   (a) $\Sigma(x, y) = x + y$
   (b) $\Pi(x, y) = xy$
   (c) $Exp(x, y) = x^y$
   (d) $Fac(x) = x!$
   (e) $Pd(x) = \begin{cases} x - 1 & \text{if } x \geq 1 \\ 0 & \text{if } x = 0 \end{cases}$
   (f) ${}^\circ d(x, y) = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$
   (g) ${}^\circ D(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$
   (h) $k(x, y) = |x - y|$
   (i) $E(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$
3. Summation and Products
   (a) Let $f^{(2)} : \mathbb{N}_0^2 \to \mathbb{N}_0$. We define $F^{(2)}$ and $G^{(2)}$ by
   $$F(x, y) = \sum_{k=0}^{y} f(x, k)$$
   $$G(x, y) = \prod_{k=0}^{y} f(x, k)$$
   Show that F and G are in **PRF**.
   (b) More general, let $f^{(k+1)} \in$ **PRF**. We define $F^{(k+1)}$ and $G^{(k+1)}$ by
   $$F(X, y) = \sum_{k=0}^{y} f(X, k)$$
   $$G(X, y) = \prod_{k=0}^{y} f(X, k)$$
   where X is a k-tuple. Show that F and G are in **PRF**.
4. Let $f^{(1)} \in$ **PRF**. We define a new function $F^{(2)}$, called *power function of f*, as
   $$F(x, y) = (\underbrace{f \circ f \circ \cdots \circ f}_{y \text{ times}})(x)$$
   or more formally,
   $$F(x, y) = \begin{cases} x & \text{if } y = 0 \\ (f \circ F)(x, y - 1) & \text{if } y > 0 \end{cases}$$
   Notation: $F(x, y) = f^y(x)$.
   (a) Show that $\Sigma(x, y) = s^y(x)$.
   (b) Show that if $f \in$ **PRF**, then $F \in$ **PRF**.
   (c) Write the function ${}^\circ d$ using the power function.

Primitive recursive sets

Reminder: Let $k \in \mathbb{N}$. A subset $A \subseteq \mathbb{N}_0^k$ is said to be a primitive recursive set ($A \in \mathbf{PRS}$) if its characteristic function $\chi_A : \mathbb{N}_0^k \to \mathbb{N}$ is primitive recursive.

5. Show that every unitary subset of $\mathbb{N}_0$ is in **PRS**.

6. Show that if $A, B \subseteq \mathbb{N}_0$ are in **PRS**, then $A \cup B$, $A \cap B$ and $\mathbb{N}_0 \setminus A$ are in **PRS**.

7. Show that every finite subset of $\mathbb{N}_0$ is in **PRS**.

8. Repeat the previous three exercises considering subsets of $\mathbb{N}_0^k$ with $k \in \mathbb{N}$.

9. Show that the set of even numbers is in **PRS**.

10. Show that the set of numbers multiple of 3 is in **PRS**.

    Tip: Show that the function $r_3 : \mathbb{N}_0 \to \mathbb{N}_0$ which takes a natural number and outputs the rest of its division by 3 is in **PRF**. Then write the characteristic function of the set of multiple of 3 in terms of r_3.

Primitive recursive relations

Reminder: A relation $R \subseteq \mathbb{N}_0 \times \mathbb{N}_0$ is said to be a primitive recursive relation ($R \in \mathbf{PRR}$) if it is in **PRS**.

11. Show that $=$, $\neq$, $\leq$ and $>$ are in **PRR**.

12. Prove that if $R, S \in \mathbf{PRR}$, then the following relations are also in **PRR**

    (a) $xTy = xRy \wedge xSy$

    (b) $xUy = xRy \vee xSy$

    (c) $x(\neg R)y = \neg(xRy)$

13. Looking at the last exercises, is there any other way to prove $=, > \in \mathbf{PRR}$?

14. Let $R \in \mathbb{N}_0 \times \mathbb{N}_0$. We define $\bigwedge R$ and $\bigvee R$ as follows.

$$x(\bigwedge R)y = \forall k \in \mathbb{N}_0 \bullet 0 \leq k \leq y \Rightarrow xRk$$

$$x(\bigvee R)y = \exists k \in \mathbb{N}_0 \bullet 0 \leq k \leq y \wedge xRk$$

    Show that if R is in **PRR**, then $\bigwedge R$ and $\bigvee R$ are also in **PRR**.

Extra

15. Show that the following function is primitive recursive.

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is multiple of 3} \\ x+3 & \text{if } x \text{ has a rest of 1 when dividing by 3} \\ x! & \text{if } x \text{ has a rest of 2 when dividing by 3} \end{cases}$$

16. Show that the divisibility relation between natural numbers is in **PRR**.

    Tip: Define the family of functions $r_a^{(1)}$ for $a = 1, 2, \ldots$ such that $r_a^{(1)}(n)$ outputs the rest of the division of n by a. Then write the characteristic function of the relation in terms of those functions.