Typing quantum superpositions and measurement*

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Abstract

We propose a way to unify two approaches of non-cloning in quantum lambda-calculi. The first approach is to forbid duplicating variables, while the second is to consider all lambda-terms as algebraic-linear functions. We illustrate this idea by defining a quantum extension of first-order simply-typed lambda-calculus, where the type is linear on superposition, while allows cloning base vectors.

1998 ACM Subject Classification F.3.2 Semantics of Programming Languages

Keywords and phrases quantum computing, lambda-calculus, simply types, linear logic

Digital Object Identifier 10.4230/LIPIcs.Submitted.2017.

1 Introduction

In λ-calculus, applying the term $\lambda x\ (x\otimes x)$, that expresses a non-linear function, to a term u yields the term $(\lambda x\ (x\otimes x))u$, that reduces to $u\otimes u$. But "cloning" this vector u is forbidden in quantum computing. Various quantum λ-calculi address this problem in different ways.

One way is to forbid the construction of the term $\lambda x\ (x\otimes x)$ using a typing system inspired from linear logic [1,9], leading to logic-linear calculi [2,10,13,15,17]. Another is to consider that all λ-terms express linear functions. The term $\lambda x\ (x\otimes x)$, for instance, expresses the linear function that maps $|0\rangle$ to $|0\rangle\otimes|0\rangle$ and $|1\rangle$ to $|1\rangle\otimes|1\rangle$. This leads to restrict beta-reduction to the case where u is a base vector (in the computational basis) and to add the linearity rule $f(u+v)\longrightarrow(fu+fv)$, leading to algebraic-linear calculi [3–6,8].

Each solution has its advantages and drawbacks. Interpreting λ-terms as algebraic-linear functions permits to reduce the term $(\lambda x\ x?|0\rangle{\cdot}|1\rangle)(\alpha.\,|0\rangle+\beta.\,|1\rangle)$ to $(\alpha.(\lambda x\ x?|0\rangle{\cdot}|1\rangle)|0\rangle+\beta.(\lambda x\ x?|0\rangle{\cdot}|1\rangle)|1\rangle)$ then to $(\alpha.\,|1\rangle+\beta.\,|0\rangle)$, instead of reducing it to $(\alpha.\,|0\rangle+\beta.\,|1\rangle)?|0\rangle{\cdot}|1\rangle$ that would be blocked. This explains that this linearity rule, that is systematic in the algebraic-linear languages cited above, is also present for the condition (the so-called $\mathbf{if}^\circ$ operator) in [2].

However, interpreting all λ-terms as linear functions forbids to extend the calculus with non-linear operators, such as measurement. For instance, the term $(\lambda x\ \pi x)(|0\rangle+|1\rangle)$, where π represents a measurement in the computational basis, would reduce to $((\lambda x\ \pi x)\,|0\rangle+(\lambda x\ \pi x)\,|1\rangle)$, while it should reduce to $|0\rangle$ with probability $\frac{1}{2}$ and to $|1\rangle$ with probability $\frac{1}{2}$.

In this paper, we propose a way to unify the two approaches, distinguishing duplicable and non-duplicable data by their type, like in the logic-linear calculi; and interpreting λ-terms as linear functions, like in the algebraic-linear calculi, when they expect duplicable data. We illustrate this idea with an example of such a calculus.

* Partially funded by the STIC-AmSud Project FoQCoSS and PICT-PRH 2015-1208.

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Submitted.
Editors: John Q. Open and Joan R. Acces; Article No. ; pp. :1–:38
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In this calculus, a qubit has type $\mathbb{Q}$ when it is in the computational basis, hence duplicable (non-linear), and $S(\mathbb{Q})$ when it is a superposition, hence non-duplicable (linear). Hence, the term $|0\rangle \otimes (|0\rangle + |1\rangle)$ has type $\mathbb{Q} \otimes S(\mathbb{Q})$. Giving this type to this term and the type $S(\mathbb{Q} \otimes \mathbb{Q})$ to the term $(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle)$ however jeopardizes the subject reduction property as, using the bilinearity of the product, the former should develop to the latter. This dilemma is not specific to quantum computing as computing is often a non-reversible process where some information is lost. For instance, if we express, in its type, that the term $(X-1)(X-2)$ is a product of two polynomials, developing it to $X^2 - 3X + 2$ does not preserve this type. A solution is to introduce, in the language, an explicit cast. For example, from the type of tensor products to the type of arbitrary vectors. The term $|0\rangle \otimes (|0\rangle + |1\rangle)$ then has type $\mathbb{Q} \otimes S(\mathbb{Q})$ and it cannot be reduced. But the term $\Uparrow (|0\rangle \otimes (|0\rangle + |1\rangle))$ has type $S(\mathbb{Q} \otimes \mathbb{Q})$ and can be developed to $(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle)$.

This language permits expressing quantum algorithms with a very precise information about the nature of the data processed by these algorithms.

Outline of the paper

In Section 2 we introduce the calculus, without tensor. In Section 3 we extend the language with a tensor operator for multiple-qubits systems. In Section 4 we prove that, thanks to the explicit casts, the resulting system indeed has the Subject Reduction property. Finally, in Section 5 we express two non-trivial examples in our calculus: the Deutsch algorithm and the Teleportation algorithm, demonstrating the expressivity of the proposed language.

2 No cloning, superpositions and measurement

The grammar of types is

$$\Psi := \mathbb{Q} \mid S(\Psi) \qquad \text{Qubit types } (\Psi)$$
$$A := \Psi \mid \Psi \Rightarrow A \mid S(A) \qquad \text{Types } (\mathcal{T})$$

and that of terms

$$b := x \mid \lambda x : \Psi\ t \mid\ |0\rangle \ \mid\ |1\rangle \qquad \text{Base terms } (\mathcal{B})$$
$$v := b \mid (v+v) \mid \vec{0}_{S(A)} \mid \alpha.v \qquad \text{Values } (\mathcal{V})$$
$$t := v \mid tt \mid (t+t) \mid \pi t \mid ?\cdot \mid \alpha.t \qquad \text{Terms } (\Lambda)$$

where $\alpha \in \mathbb{C}$.

Terms are variables, abstractions, applications, two constants for base qubits ($|0\rangle$ and $|1\rangle$), linear combinations of terms (built with addition and product by a scalar, addition being commutative and associative), a family of constants for the null vectors, one for each type of the form $S(A)$, ($\vec{0}_{S(A)}$), and an if-then-else construction ($?\cdot$) deciding on base vectors. We also include a symbol π for measurement in the computational basis.

The grammar is split into base terms (non-superposed values), general values, and general terms. Types are also split into qubit types and general types.

The set of free variables of a term t is defined as usual in λ-calculus and denoted by $FV(t)$. We use $[\alpha.]t$ as a notation to refer indistinctly to $\alpha.t$ and to t. We use $-t$ as a shorthand notation for $-1.t$, and $(t-r)$ as a shorthand notation for $(t + (-r))$. The term $(t-t)$ has type $S(A)$, and reduces to $\vec{0}_{S(A)}$, which is not a base term.

An important property of this calculus is that types $S(\cdot)$ are linear types. Indeed, those correspond to superpositions, and so no duplication is allowed on them. Instead, at this

$$\frac{}{x:\Psi\vdash x:\Psi}\ Ax \qquad \frac{}{\vdash\vec{0}_{S(A)}:S(A)}\ Ax_{\vec 0} \qquad \frac{}{\vdash|0\rangle:\mathbb{Q}}\ Ax_{|0\rangle} \qquad \frac{}{\vdash|1\rangle:\mathbb{Q}}\ Ax_{|1\rangle}$$

$$\frac{\Gamma\vdash t:A}{\Gamma\vdash\alpha.t:S(A)}\ S_I^\alpha \qquad \frac{\Gamma\vdash t:A\quad \Delta\vdash u:A}{\Gamma,\Delta\vdash(t+u):S(A)}\ S_I^+ \qquad \frac{\Gamma\vdash t:S(\mathbb{Q})}{\Gamma\vdash\pi t:\mathbb{Q}}\ S_E$$

$$\frac{\Gamma\vdash t:A\ {\scriptstyle(A\preceq B)}}{\Gamma\vdash t:B}\ \preceq \qquad \frac{}{\vdash ?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\ \mathit{If} \qquad \frac{\Gamma,x:\Psi\vdash t:A}{\Gamma\vdash\lambda x:\Psi\ t:\Psi\Rightarrow A}\ \Rightarrow_I$$

$$\frac{\Gamma\vdash t:\Psi\Rightarrow A\quad \Delta\vdash u:\Psi}{\Gamma,\Delta\vdash tu:A}\ \Rightarrow_E \qquad \frac{\Gamma\vdash t:S(\Psi\Rightarrow A)\quad \Delta\vdash u:S(\Psi)}{\Gamma,\Delta\vdash tu:S(A)}\ \Rightarrow_{ES}$$

$$\frac{\Gamma\vdash t:A}{\Gamma,x:\mathbb{Q}\vdash t:A}\ W \qquad \frac{\Gamma,x:\mathbb{Q},y:\mathbb{Q}\vdash t:A}{\Gamma,x:\mathbb{Q}\vdash(x/y)t:A}\ C$$

Contexts Γ and Δ have a disjoint support.

Figure 1 Typing rules

tensor-free stage, a type without an $S(\cdot)$ on head position is a non-linear type, such as $\mathbb{Q}$, which correspond to base terms, i.e. terms that can be cloned. A non-linear function is allowed to be applied to a linear argument, for example, $\lambda x:\mathbb{Q}\ (fxx)$ can be applied to $(\frac{1}{\sqrt{2}}.|0\rangle+\frac{1}{\sqrt{2}}.|1\rangle)$, however, it distributes in the following way:

$$\begin{aligned}&(\lambda x:\mathbb{Q}\ (fxx))\ (\frac{1}{\sqrt{2}}.|0\rangle+\frac{1}{\sqrt{2}}.|1\rangle)\\ &\longrightarrow(\frac{1}{\sqrt{2}}.(\lambda x:\mathbb{Q}\ (fxx))\,|0\rangle+\frac{1}{\sqrt{2}}.(\lambda x:\mathbb{Q}\ (fxx))\,|1\rangle)\\ &\longrightarrow(\frac{1}{\sqrt{2}}.(f\,|0\rangle\,|0\rangle)+\frac{1}{\sqrt{2}}.(f\,|1\rangle\,|1\rangle))\end{aligned}$$

Hence, the beta reduction occurs only when the type of the argument is the same as the type expected by the abstraction. Thus, the rewrite system depends on types. For this reason, we describe first the type system, and only then the rewrite system.

A type A is informally interpreted as a set of vectors and $S(A)$ is the vector space generated by such a set. Hence, we naturally have $A\subseteq S(A)$ and $S(S(A))=S(A)$. Therefore, we define the following subtyping relation on types.

▶ **Definition 1** (Subtyping)**.** The relation $\preceq$ is a preorder defined by

$$\frac{}{A\preceq S(A)} \qquad \frac{}{S(S(A))\preceq S(A)} \qquad \frac{A\preceq B}{\Psi\Rightarrow A\preceq\Psi\Rightarrow B} \qquad \frac{A\preceq B}{S(A)\preceq S(B)}$$

The type system is given in Figure 1 and the rewrite system in Figure 2.

In the type system, rule Ax allows typing variables only with qubit types. Hence, the system is first-order and only qubits can be passed as arguments (see Section 6). Rule $Ax_{\vec 0}$ types the null vector as a non-base term, because the null vector cannot belong to the base of any vector space. Rules $Ax_{|0\rangle}$ and $Ax_{|1\rangle}$ type the base qubits with the base type $\mathbb{Q}$.

Rule $\preceq$ is the subsumption rule, allowing to type a term with a more general type. For example, the term $|0\rangle$ has type $\mathbb{Q}$ and also the more general type $S(\mathbb{Q})$. Note that $((|0\rangle+|0\rangle)-|0\rangle)$ has type $S(\mathbb{Q})$ and reduces to $|0\rangle$ that has the same type $S(\mathbb{Q})$. Reducing this term to $|0\rangle$ of type $\mathbb{Q}$ would not preserve its type. Moreover, this type would contain

Beta rules

If b has type $\mathbb{Q}$ and $b \in \mathcal{B}$, then

$(\lambda x : \mathbb{Q}\ t)b$	$\longrightarrow_{(1)}$	$(b/x)t$	(β_{b})

If u has type $S(\Psi)$, then

$(\lambda x : S(\Psi)\ t)u$	$\longrightarrow_{(1)}$	$(u/x)t$	(β_{n})

If-then-else

$\|1\rangle ?u{\cdot}v$	$\longrightarrow_{(1)}$	u	(if_1)
$\|0\rangle ?u{\cdot}v$	$\longrightarrow_{(1)}$	v	(if_0)

Linear distribution

If t has type $\mathbb{Q} \Rightarrow A$, then

$t(u+v)$	$\longrightarrow_{(1)}$	$(tu+tv)$	$(\mathsf{lin}_{\mathsf{r}}^{+})$

If t has type $\mathbb{Q} \Rightarrow A$ then

$t(\alpha.u)$	$\longrightarrow_{(1)}$	$\alpha.tu$	$(\mathsf{lin}_{\mathsf{r}}^{\alpha})$

If t has type $\mathbb{Q} \Rightarrow A$, then

$t\vec{0}_{S(\mathbb{Q})}$	$\longrightarrow_{(1)}$	$\vec{0}_{S(A)}$	$(\mathsf{lin}_{\mathsf{r}}^{0})$
$(t+u)v$	$\longrightarrow_{(1)}$	$(tv+uv)$	$(\mathsf{lin}_{\mathsf{l}}^{+})$
$(\alpha.t)u$	$\longrightarrow_{(1)}$	$\alpha.tu$	$(\mathsf{lin}_{\mathsf{l}}^{\alpha})$
$\vec{0}_{S(\mathbb{Q}\Rightarrow A)}t$	$\longrightarrow_{(1)}$	$\vec{0}_{S(A)}$	$(\mathsf{lin}_{\mathsf{l}}^{0})$

Vector space axioms

$(\vec{0}_{S(A)} + t)$	$\longrightarrow_{(1)}$	t	(neutral)
$1.t$	$\longrightarrow_{(1)}$	t	(unit)

If t has type A, then

$0.t$	$\longrightarrow_{(1)}$	$\vec{0}_{S(A)}$	(zero_{α})
$\alpha.\vec{0}_{S(A)}$	$\longrightarrow_{(1)}$	$\vec{0}_{S(A)}$	(zero)
$\alpha.(\beta.t)$	$\longrightarrow_{(1)}$	$(\alpha\times\beta).t$	(prod)
$\alpha.(t+u)$	$\longrightarrow_{(1)}$	$(\alpha.t+\alpha.u)$	(αdist)
$(\alpha.t+\beta.t)$	$\longrightarrow_{(1)}$	$(\alpha+\beta).t$	(fact)
$(\alpha.t+t)$	$\longrightarrow_{(1)}$	$(\alpha+1).t$	(fact^1)
$(t+t)$	$\longrightarrow_{(1)}$	$2.t$	(fact^2)

Modulo AC

$(u+v)$	$=_{AC}$	$(v+u)$	(comm)
$((u+v)+w)$	$=_{AC}$	$(u+(v+w))$	(assoc)

Projection

$\pi(\sum_{i=1}^{n}[\alpha_i.]b_i)$	$\longrightarrow$	$\left(\frac{\|\alpha_k\|^2}{\sum_{i=1}^{n}\|\alpha_i\|^2}\right) b_k$	(proj)

where, in rule (proj), $\forall i$, $b_i = |0\rangle$ or $b_i = |1\rangle$, $\sum_{i=1}^{n}\alpha_i.b_i$ is normal (so $1 \le n \le 2$), if an α_k is absent, $|\alpha_k|^2 = 1$, and $1 \le k \le n$.

Figure 2 Reduction rules

information impossible to compute, because the value $|0\rangle$ is not the result of a measurement, but of an interference.

Rule S_I^{α} states that a term multiplied by a scalar is not a base term. Even if the scalar is just a phase, we must type the term with an $S(\cdot)$ type, because our projector removes the scalars, so having the scalar means that it has not been measured yet. Rule S_I^{+} is the analog for sums to the previous rule. Rule S_E is the elimination of the superposition, which is achieved by measuring (using the π operator).

Rule *If* types the if-then-else construction. We use $r?s{\cdot}t$ as a notation for $(?{\cdot})rst$. Notice that it is typed as a non-linear function, and so, the if-then-else linearly distributes over superpositions. For example: $(\alpha.|0\rangle + \beta.|1\rangle)?t{\cdot}r \to^* (\alpha.|0\rangle?t{\cdot}r + \beta.|1\rangle?t{\cdot}r) \to^* (\alpha.t + \beta.r)$. This way, we avoid the if-then-else construction from measuring its argument.

Rule $\Rightarrow_I$ is standard. Rule $\Rightarrow_E$ is standard for linear type systems. Rule $\Rightarrow_{ES}$ is the elimination for superpositions, corresponding to the linear distribution. Notice that the type of the argument is a superposition of the argument expected by the abstraction ($S(\Psi)$ vs. Ψ). Also, the abstraction is allowed to be a superposition. If, for example, we want to apply the sum of functions $(f+g)$ to the base argument $|0\rangle$, we would obtain the superposition $(f\,|0\rangle + g\,|0\rangle)$. The typing is as follows:

$$\dfrac{\dfrac{\vdash f : \mathbb{Q} \Rightarrow A \quad \vdash g : \mathbb{Q} \Rightarrow A}{\vdash (f+g) : S(\mathbb{Q} \Rightarrow A)}\, S_I^{+} \qquad \dfrac{\dfrac{}{\vdash |0\rangle : \mathbb{Q}}\, Ax_{|0\rangle}}{\vdash |0\rangle : S(\mathbb{Q})}\, \preceq}{\vdash (f+g)\,|0\rangle : S(A)}\, \Rightarrow_{ES}$$

which reduces to

$$\dfrac{\dfrac{\vdash f:\mathbb{Q}\Rightarrow A \quad \dfrac{}{\vdash |0\rangle:\mathbb{Q}}Ax_{|0\rangle}}{\vdash f\,|0\rangle : A}\Rightarrow_E \quad \dfrac{\vdash g:\mathbb{Q}\Rightarrow A \quad \dfrac{}{\vdash |0\rangle:\mathbb{Q}}Ax_{|0\rangle}}{\vdash g\,|0\rangle : A}\Rightarrow_E}{\vdash (f\,|0\rangle + g\,|0\rangle) : S(A)}S_I^+$$

Similarly, a linear function ($\vdash f:\mathbb{Q}\Rightarrow A$) applied to a superposition ($|0\rangle+|1\rangle$) reduces to a superposition ($f\,|0\rangle + f\,|1\rangle$). The typing is as follows:

$$\dfrac{\dfrac{\vdash f:\mathbb{Q}\Rightarrow A}{\vdash f:S(\mathbb{Q}\Rightarrow A)}\preceq \quad \dfrac{\dfrac{}{\vdash |0\rangle:\mathbb{Q}}Ax_{|0\rangle} \quad \dfrac{}{\vdash |1\rangle:\mathbb{Q}}Ax_{|1\rangle}}{\vdash (|0\rangle+|1\rangle):S(\mathbb{Q})}S_I^+}{\vdash f(|0\rangle+|1\rangle):S(A)}\Rightarrow_{ES}$$

which reduces to

$$\dfrac{\dfrac{\vdash f:\mathbb{Q}\Rightarrow A \quad \dfrac{}{\vdash |0\rangle:\mathbb{Q}}Ax_{|0\rangle}}{\vdash f\,|0\rangle : A}\Rightarrow_E \quad \dfrac{\vdash f:\mathbb{Q}\Rightarrow A \quad \dfrac{}{\vdash |1\rangle:\mathbb{Q}}Ax_{|1\rangle}}{\vdash f\,|1\rangle : A}\Rightarrow_E}{\vdash (f\,|0\rangle + f\,|1\rangle) : S(A)}S_I^+$$

Finally, Rules W and C correspond to weakening and contraction on variables with base types. The rationale is that base terms can be cloned.

In the rewrite system, the relation $\longrightarrow_{(p)}$ is a probabilistic relation where p is the probability of occurrence. Every rewrite rule have a probability 1 of occurrence, except for the projection ((proj) rule). The rewrite system depends on the typing, in particular an abstraction can either expect a base term as argument (that is, a non-linear term) or a superposition, which has to be treated linearly. However, an abstraction expecting a non-linear argument can be given a superposition (which is linear), and it is typable, only that the reduction distributes before beta-reduction.

There are two beta rules. Rule (β_{b}) acts only when the argument is a base term, and the type expected by the abstraction is a base type. Hence, rule (β_{b}) is "call-by-base" (base terms coincides with values of λ-calculus, while values on this calculus also includes superpositions of base terms and the null vector). Instead, (β_{n}) is the usual call-by-name beta rule. They are distinguished by the type of the argument. Rule (β_{b}) acts on non-linear functions while (β_{n}) is for linear functions. The test on the type of the argument is due to the type system that allows an argument with a type not matching with the type expected by the abstraction (in such a case, one of the linear distribution rules applies).

Since there are two beta reductions, the contextual rule admitting reducing the argument on an application is valid only when the abstraction expects an argument of type $\mathbb{Q}$, that is, $(\lambda x:\mathbb{Q}\ v)t \longrightarrow_{(p)} (\lambda x:\mathbb{Q}\ v)u$ when $t\longrightarrow_{(p)} u$. If the argument is typed with a base type, then it reduces to a term that can be cloned, and we must reduce it first to ensure that we are cloning a term that can be cloned. For example, a measure over a superposition (e.g. $\pi(|0\rangle+|1\rangle)$) has a base type $\mathbb{Q}$, but it cannot be cloned until it is reduced. Indeed, $(\lambda x:\mathbb{Q}\ (fxx))\pi(|0\rangle+|1\rangle)$ can reduce either to $f\,|0\rangle\,|0\rangle$ or $f\,|1\rangle\,|1\rangle$, but never to $f\,|0\rangle\,|1\rangle$ or $f\,|1\rangle\,|0\rangle$, which would be possible only if the measure happens after the cloning machine.

The group If-then-else contains the tests over the base qubits $|0\rangle$ and $|1\rangle$.

Linear distribution rules. The first three rules (marked with subindex r), are the rules that are used when a non-linear abstraction is applied to a linear argument (that is, when an abstraction expecting a base term is given a superposition). In these cases the beta reductions cannot be used since the side conditions on types are not met. Hence, this distributivity rules applies instead.

For example, let us give more details in the reduction sequence on the example given at the beginning of this Section.

$$
\begin{aligned}
&(\lambda x:\mathbb{Q}\ (fxx))(\frac{1}{\sqrt{2}}.\,|0\rangle + \frac{1}{\sqrt{2}}.\,|1\rangle)\\
&\xrightarrow{(\mathsf{lin}^+_\mathsf{r})}_{(1)}\ ((\lambda x:\mathbb{Q}\ (fxx))\frac{1}{\sqrt{2}}.\,|0\rangle + (\lambda x:\mathbb{Q}\ (fxx))\frac{1}{\sqrt{2}}.\,|1\rangle)\\
&\xrightarrow{(\mathsf{lin}^\alpha_\mathsf{r})^2}_{(1)}\ (\frac{1}{\sqrt{2}}.(\lambda x:\mathbb{Q}\ (fxx))\,|0\rangle + \frac{1}{\sqrt{2}}.(\lambda x:\mathbb{Q}\ (fxx))\,|1\rangle)\\
&\xrightarrow{(\beta_\mathsf{b})^2}_{(1)}\ (\frac{1}{\sqrt{2}}.f\,|0\rangle\,|0\rangle + \frac{1}{\sqrt{2}}.f\,|1\rangle\,|1\rangle)
\end{aligned}
$$

The remaining rules in this group deal with a superposition of functions. For example, rule $(\mathsf{lin}^+_\mathsf{l})$ is the sum of functions: A superposition is a sum, therefore, if an argument is given to a sum of functions, it needs to be given to each function in the sum. We use a weak reduction strategy (i.e. reduction occurs only on closed terms), hence the argument v on this rule is closed, otherwise, it could not be typed. For example

$$x : S(\mathbb{Q}), t : \mathbb{Q} \Rightarrow \mathbb{Q}, u : \mathbb{Q} \Rightarrow \mathbb{Q} \vdash (t+u)x : S(\mathbb{Q})$$

is derivable, but

$$x : S(\mathbb{Q}), t : \mathbb{Q} \Rightarrow \mathbb{Q}, u : \mathbb{Q} \Rightarrow \mathbb{Q} \vdash (tx+ux) : S(\mathbb{Q})$$

is not.

The Vector space axioms rules are the directed axioms of vector spaces [5,6]. The Modulo AC rules are not proper rewrite rules, but express that we consider the symbol $+$ to be associative and commutative, and hence our rewrite system is *rewrite modulo AC* [14].

Finally, rule (proj) is the projection over weighted associative pairs, that is, the projection over a generalization of multisets where the multiplicities are given by complex numbers. This reduction rule is the only one with a probability different from 1, and it is given by the square of the modulus of the weights[1], implementing this way the quantum measurement.

Remark, to conclude, that this calculus can represent only pure states, and not mixed states. For example, let $|+\rangle = \frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle)$. The terms $(\lambda x : S(\mathbb{Q})\ (\lambda y:\mathbb{Q}\ y?(\mathsf{Z}x){\cdot}x)(\pi_1\,|+\rangle))\,|+\rangle$ and $(\lambda x : S(\mathbb{Q})\ \pi_1(x))\,|+\rangle$ may be considered equivalent if taking into account the density matrix representation of mixed states. Indeed, the first reduces either to $|+\rangle$ or $|-\rangle$, with probability $\frac{1}{2}$ each, while the second reduces to $|0\rangle$ or to $|1\rangle$, with probability $\frac{1}{2}$ each. The sets of pure states $\{(\frac{1}{2},|+\rangle),(\frac{1}{2},|-\rangle)\}$ and $\{(\frac{1}{2},|0\rangle),(\frac{1}{2},|1\rangle)\}$ have both density matrix $\frac{I}{2}$, and hence are indistinguishable. However, once the result of the measure is known, the pure states can be distinguished.

3 Multi-qubit systems: Tensor products

A multi-qubit system is represented with the tensor product between single-qubit Hilbert spaces. The tensor product of terms can be seen as an ordered list. Hence, we represent the

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[1] We speak about weights and not amplitudes, since the vector may not have norm 1. The projection rule normalizes the vector while reducing.

tensor product as a conjunction-like operator. The distributivity of linear combinations over tensor products is not trivially tracked in the type system, and so an explicit cast between types is also added.

Each level in the term grammar (base terms, values and general terms) is extended with the tensor of the terms in such a level. The primitives *head* and *tail* are added to the general terms. The projector π is generalized to π_j, where the subindex j stands for the number of qubits to be measured, which are those in the first j positions.

Notice that it is always possible to do a swap between qubits and so place the qubits to be measured at the beginning. For instance, $\lambda x : \mathbb{Q} \otimes \mathbb{Q}\ (\mathit{tail}\ x \otimes \mathit{head}\ x)$.

An explicit type cast of a term t ($\Uparrow_{S(A)}^{S(B\otimes C)} t$) is included in the general terms. It is only allowed to cast a superposed type into a superposed tensor product. We also add the tensor between types, and, as a consequence, a new level. Indeed, without tensors, the only base qubit type was $\mathbb{Q}$. With tensor, we need to put them in a new level where also tensor of base qubits are considered as base qubits.

$$\begin{aligned}
Q &:= \mathbb{Q} \mid Q \otimes Q && \text{Base qubit types } (\mathcal{B})\\
\Psi &:= Q \mid S(\Psi) \mid \Psi \otimes \Psi && \text{Qubit types } (\Psi)\\
A &:= \Psi \mid \Psi \Rightarrow A \mid S(A) \mid A \otimes A && \text{Types } (\mathcal{T})
\end{aligned}$$

$$\begin{aligned}
b &:= x \mid \lambda x : \Psi\ t \mid\ |0\rangle\ \mid\ |1\rangle\ \mid b \otimes b && \text{Base terms } (\mathcal{B})\\
v &:= b \mid (v+v) \mid \vec{0}_{S(A)} \mid \alpha.v \mid v \otimes v && \text{Values } (\mathcal{V})\\
t &:= v \mid tt \mid (t+t) \mid \pi_j t \mid ?\cdot \mid \alpha.t && \text{Terms } (\Lambda)\\
&\quad \mid t \otimes t \mid \mathit{head}\ t \mid \mathit{tail}\ t \mid \Uparrow_{S(A)}^{S(B\otimes C)} t
\end{aligned}$$

where $\alpha \in \mathbb{C}$

The type system includes all the typing rules given in Figure 1, plus the rules for tensor, for cast, and an updated rule S_E, for which we introduce the following notation:

▶ **Definition 2.** Let $S \subseteq \{1, \cdots, n\}$. We define Q_n^S inductively by:

$$\begin{aligned}
Q_n^S &= \begin{cases} A_{n-1}^S(\mathbb{Q}) & \text{if } n \notin S\\ A_{n-1}^{S\setminus\{n\}}(S(\mathbb{Q})) & \text{if } n \in S\end{cases}\\
A_0^{\emptyset}(B) &= B\\
A_{k+1}^S(\mathbb{Q}) &= \begin{cases} A_k^S(\mathbb{Q}) \otimes \mathbb{Q} & \text{if } k+1 \notin S\\ A_k^{S\setminus\{k+1\}}(S(\mathbb{Q})) \otimes \mathbb{Q} & \text{if } k+1 \in S\end{cases}\\
A_{k+1}^S(S(B)) &= \begin{cases} A_k^S(\mathbb{Q}) \otimes S(B) & \text{if } k+1 \notin S\\ A_k^{S\setminus\{k+1\}}(S(\mathbb{Q} \otimes B)) & \text{if } k+1 \in S\end{cases}
\end{aligned}$$

where B is any type.

In simple words, notation Q_n^S stands for a tensor of n qubits, where those indexed by the set S are superposed and typed with the most general type, for example $Q_3^{\{1,2\}}$ stands for

$$\frac{}{x:\Psi\vdash x:\Psi}\ Ax \qquad \frac{}{\vdash \vec{0}_{S(A)}:S(A)}\ Ax_{\vec{0}} \qquad \frac{}{\vdash |0\rangle:\mathbb{Q}}\ Ax_{|0\rangle} \qquad \frac{}{\vdash |1\rangle:\mathbb{Q}}\ Ax_{|1\rangle}$$

$$\frac{\Gamma\vdash t:A}{\Gamma\vdash \alpha.t:S(A)}\ S_I^{\alpha} \qquad \frac{\Gamma\vdash t:A \quad \Delta\vdash u:A}{\Gamma,\Delta\vdash (t+u):S(A)}\ S_I^{+} \qquad \frac{\Gamma\vdash t:Q_n^S}{\Gamma\vdash \pi_j t:Q_n^{S\setminus\{1,\dots,j\}}}\ S_E\ \left({}^{S\subseteq\mathbb{N}^{\leq n}}_{j\leq n}\right)$$

$$\frac{\Gamma\vdash t:A\ (A\preceq B)}{\Gamma\vdash t:B}\ \preceq \qquad \frac{}{\vdash ?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\ \mathit{If} \qquad \frac{\Gamma,x:\Psi\vdash t:A}{\Gamma\vdash \lambda x:\Psi\ t:\Psi\Rightarrow A}\ \Rightarrow_I$$

$$\frac{\Gamma\vdash t:\Psi\Rightarrow A \quad \Delta\vdash u:\Psi}{\Gamma,\Delta\vdash tu:A}\ \Rightarrow_E \qquad \frac{\Gamma\vdash t:S(\Psi\Rightarrow A) \quad \Delta\vdash u:S(\Psi)}{\Gamma,\Delta\vdash tu:S(A)}\ \Rightarrow_{ES}$$

$$\frac{\Gamma\vdash t:A}{\Gamma,x:Q\vdash t:A}\ W \qquad \frac{\Gamma,x:Q,y:Q\vdash t:A}{\Gamma,x:Q\vdash (x/y)t:A}\ C$$

$$\frac{\Gamma\vdash t:A \quad \Delta\vdash u:B}{\Gamma,\Delta\vdash t\otimes u:A\otimes B}\ \otimes_I \qquad \frac{\Gamma\vdash t:\mathbb{Q}\otimes Q}{\Gamma\vdash \mathit{head}\ t:\mathbb{Q}}\ \otimes_{Er} \qquad \frac{\Gamma\vdash t:\mathbb{Q}\otimes Q}{\Gamma\vdash \mathit{tail}\ t:Q}\ \otimes_{El}$$

$$\frac{\Gamma\vdash t:S(S(A)\otimes B)}{\Gamma\vdash \Uparrow_{S(S(A)\otimes B)}^{S(A\otimes B)}\ t:S(A\otimes B)}\ \Uparrow_r \qquad \frac{\Gamma\vdash t:S(A\otimes S(B))}{\Gamma\vdash \Uparrow_{S(A\otimes S(B))}^{S(A\otimes B)}\ t:S(A\otimes B)}\ \Uparrow_l$$

$$\frac{\Gamma\vdash \Uparrow_{S(B)}^{S(A)}\ t:S(A)}{\Gamma\vdash \Uparrow_{S(B)}^{S(A)}\ \alpha.t:S(A)}\ \Uparrow^{\alpha} \qquad \frac{\Gamma\vdash \Uparrow_{S(B)}^{S(A)}\ t:S(A) \quad \Delta\vdash \Uparrow_{S(B)}^{S(A)}\ r:S(A)}{\Gamma,\Delta\vdash \Uparrow_{S(B)}^{S(A)}\ (t+r):S(A)}\ \Uparrow^{+}$$

Figure 3 Typing rules

$S(\mathbb{Q}\otimes\mathbb{Q})\otimes\mathbb{Q}$ and not for $S(\mathbb{Q})\otimes S(\mathbb{Q})\otimes\mathbb{Q}$. The following examples may be clarifying.

$$\begin{aligned}
&Q_5^{\{1,2,4\}}\\
&= A_4^{\{1,2,4\}}(\mathbb{Q})\\
&= A_3^{\{1,2\}}(S(\mathbb{Q}))\otimes\mathbb{Q}\\
&= A_2^{\{1,2\}}(\mathbb{Q})\otimes S(\mathbb{Q})\otimes\mathbb{Q}\\
&= A_1^{\{1\}}(S(\mathbb{Q}))\otimes\mathbb{Q}\otimes S(\mathbb{Q})\otimes\mathbb{Q}\\
&= A_0^{\emptyset}(S(\mathbb{Q}\otimes\mathbb{Q}))\otimes\mathbb{Q}\otimes S(\mathbb{Q})\otimes\mathbb{Q}\\
&= S(\mathbb{Q}\otimes\mathbb{Q})\otimes\mathbb{Q}\otimes S(\mathbb{Q})\otimes\mathbb{Q}
\end{aligned}
\qquad
\begin{aligned}
&Q_5^{\{4,5\}}\\
&= A_4^{\{4\}}(S(\mathbb{Q}))\\
&= A_3^{\emptyset}(S(\mathbb{Q}\otimes\mathbb{Q}))\\
&= A_2^{\emptyset}(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\\
&= A_1^{\emptyset}(\mathbb{Q})\otimes\mathbb{Q}\otimes S(\mathbb{Q}\otimes\mathbb{Q})\\
&= A_0^{\emptyset}(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q}\otimes\mathbb{Q})\\
&= \mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q}\otimes\mathbb{Q})
\end{aligned}$$

In addition, we update the subtyping relation, trivially, to also include tensor products:

▶ **Definition 3** (Subtyping)**.** The relation $\preceq$ is a preorder defined by

$$\frac{}{A\preceq S(A)} \qquad \frac{}{S(S(A))\preceq S(A)}$$

$$\frac{A\preceq B}{\Psi\Rightarrow A\preceq \Psi\Rightarrow B} \qquad \frac{A\preceq B}{S(A)\preceq S(B)} \qquad \frac{A\preceq B}{A\otimes C\preceq B\otimes C} \qquad \frac{A\preceq B}{C\otimes A\preceq C\otimes B}$$

The type system is updated as follows. Rules Ax, $Ax_{\vec{0}}$, $Ax_{|0\rangle}$, $Ax_{|1\rangle}$, $\preceq$, S_I^{α}, S_I^{+}, *If*, $\Rightarrow_I$, $\Rightarrow_E$ and $\Rightarrow_{ES}$ remain unchanged. Rule S_E types the generalized projection: we force the term to be measured to be typed with a type of the form Q_n^S (cf. Definition 2), and then, after measuring the first j qubits, the new type becomes $Q_n^{S\setminus\{1,\dots,j\}}$, that is, we remove the superposition mark $S(\cdot)$ from the first j types in the tensor product. Rules W and C are updated only to act on types Q instead of just $\mathbb{Q}$.

Rules $\otimes_I$, $\otimes_{E_r}$, $\otimes_{E_l}$ are the standard introduction and eliminations for lists. Rules $\Uparrow_r$ and $\Uparrow_l$ type the castings. Indeed, $\Uparrow_{S(A)}^{S(B)}\ t$ indicates that the term t has type $S(A)$ and will be

Beta rules

If b has type Q and $b \in \mathcal{B}$, then

$$(\lambda x : Q\ t)b \longrightarrow_{(1)} (b/x)t \qquad (\beta_{\mathsf{b}})$$

If u has type Ψ, then

$$(\lambda x : \Psi\ t)u \longrightarrow_{(1)} (u/x)t \qquad (\beta_{\mathsf{n}})$$

Vector space axioms

$$(\vec{0}_{S(A)} + t) \longrightarrow_{(1)} t \qquad (\mathsf{neutral})$$

$$1.t \longrightarrow_{(1)} t \qquad (\mathsf{unit})$$

If t has type A, then

$$0.t \longrightarrow_{(1)} \vec{0}_{S(A)} \qquad (\mathsf{zero}_\alpha)$$

$$\alpha.\vec{0}_{S(A)} \longrightarrow_{(1)} \vec{0}_{S(A)} \qquad (\mathsf{zero})$$

$$\alpha.(\beta.t) \longrightarrow_{(1)} (\alpha \times \beta).t \qquad (\mathsf{prod})$$

$$\alpha.(t+u) \longrightarrow_{(1)} (\alpha.t + \alpha.u) \qquad (\alpha\mathsf{dist})$$

$$(\alpha.t + \beta.t) \longrightarrow_{(1)} (\alpha+\beta).t \qquad (\mathsf{fact})$$

$$(\alpha.t + t) \longrightarrow_{(1)} (\alpha+1).t \qquad (\mathsf{fact}^1)$$

$$(t+t) \longrightarrow_{(1)} 2.t \qquad (\mathsf{fact}^2)$$

Modulo AC

$$(u+v) =_{AC} (v+u) \qquad (\mathsf{comm})$$

$$((u+v)+w) =_{AC} (u+(v+w)) \qquad (\mathsf{assoc})$$

Lists and if-then-else

If $h \neq u \otimes v$ and $h \in \mathcal{B}$, then

$$\mathit{head}\ (h \otimes t) \longrightarrow_{(1)} h \qquad (\mathsf{head})$$

If $h \neq u \otimes v$ and $h \in \mathcal{B}$, then

$$\mathit{tail}\ (h \otimes t) \longrightarrow_{(1)} t \qquad (\mathsf{tail})$$

$$|1\rangle ?u{\cdot}v \longrightarrow_{(1)} u \qquad (\mathsf{if}_1)$$

$$|0\rangle ?u{\cdot}v \longrightarrow_{(1)} v \qquad (\mathsf{if}_0)$$

Linear distribution

If t has type $Q \Rightarrow A$, then

$$t(u+v) \longrightarrow_{(1)} (tu+tv) \qquad (\mathsf{lin}^+_{\mathsf{r}})$$

If t has type $Q \Rightarrow A$, then

$$t(\alpha.u) \longrightarrow_{(1)} \alpha.tu \qquad (\mathsf{lin}^\alpha_{\mathsf{r}})$$

If t has type $Q \Rightarrow A$, then

$$t\vec{0}_{S(Q)} \longrightarrow_{(1)} \vec{0}_{S(A)} \qquad (\mathsf{lin}^0_{\mathsf{r}})$$

$$(t+u)v \longrightarrow_{(1)} (tv+uv) \qquad (\mathsf{lin}^+_{\mathsf{l}})$$

$$(\alpha.t)u \longrightarrow_{(1)} \alpha.tu \qquad (\mathsf{lin}^\alpha_{\mathsf{l}})$$

$$\vec{0}_{S(Q\Rightarrow A)}t \longrightarrow_{(1)} \vec{0}_{S(A)} \qquad (\mathsf{lin}^0_{\mathsf{l}})$$

Typing casts

$$\Uparrow^{S(A\otimes B)}_{S(S(A)\otimes B)} ((r+s)\otimes u) \longrightarrow_{(1)} (\Uparrow^{S(A\otimes B)}_{S(S(A)\otimes B)} (r\otimes u) + \Uparrow^{S(A\otimes B)}_{S(S(A)\otimes B)} (s\otimes u)) \qquad (\mathsf{dist}^+_{\mathsf{r}})$$

$$\Uparrow^{S(B\otimes A)}_{S(B\otimes S(A))} (u\otimes(r+s)) \longrightarrow_{(1)} (\Uparrow^{S(B\otimes A)}_{S(B\otimes S(A))} (u\otimes r) + \Uparrow^{S(B\otimes A)}_{S(B\otimes S(A))} (u\otimes s)) \qquad (\mathsf{dist}^+_{\mathsf{l}})$$

$$\Uparrow^{S(A\otimes B)}_{S(S(A)\otimes B)} ((\alpha.r)\otimes u) \longrightarrow_{(1)} \alpha.\ \Uparrow^{S(A\otimes B)}_{S(S(A)\otimes B)} (r\otimes u) \qquad (\mathsf{dist}^\alpha_{\mathsf{r}})$$

$$\Uparrow^{S(B\otimes A)}_{S(B\otimes S(A))} (u\otimes(\alpha.r)) \longrightarrow_{(1)} \alpha.\ \Uparrow^{S(B\otimes A)}_{S(B\otimes S(A))} (u\otimes r) \qquad (\mathsf{dist}^\alpha_{\mathsf{l}})$$

$$\Uparrow^{S(A\otimes B)}_{S(S(A)\otimes B)} (\vec{0}_{S(A)}\otimes u) \longrightarrow_{(1)} \vec{0}_{S(A\otimes B)} \qquad (\mathsf{dist}^0_{\mathsf{r}})$$

$$\Uparrow^{S(B\otimes A)}_{S(B\otimes S(A))} (u\otimes \vec{0}_{S(A)}) \longrightarrow_{(1)} \vec{0}_{S(B\otimes A)} \qquad (\mathsf{dist}^0_{\mathsf{l}})$$

$$\Uparrow^{S(B\otimes C)}_{S(A)} (t+u) \longrightarrow_{(1)} (\Uparrow^{S(B\otimes C)}_{S(A)} t + \Uparrow^{S(B\otimes C)}_{S(A)} u) \qquad (\mathsf{dist}^+_{\Uparrow})$$

$$\Uparrow^{S(B\otimes C)}_{S(A)} (\alpha.t) \longrightarrow_{(1)} \alpha.\ \Uparrow^{S(B\otimes C)}_{S(A)} t \qquad (\mathsf{dist}^\alpha_{\Uparrow})$$

If $u \in \mathcal{B}$, then, $\Uparrow^{S(A\otimes B)}_{S(S(A)\otimes B)} (u\otimes v) \longrightarrow_{(1)} u\otimes v$ $\qquad (\mathsf{neut}^{\Uparrow}_{\mathsf{r}})$

If $u \in \mathcal{B}$, then, $\Uparrow^{S(A\otimes B)}_{S(A\otimes S(B))} (v\otimes u) \longrightarrow_{(1)} v\otimes u$ $\qquad (\mathsf{neut}^{\Uparrow}_{\mathsf{l}})$

Projection

$$\pi_j\Big(\sum_{i=1}^{n}[\alpha_i.](b_{1i}\otimes\cdots\otimes b_{mi})\Big) \longrightarrow_{(p)} \bigotimes_{h=1}^{j} b_{hk} \otimes \sum_{i\in P}\left(\frac{\alpha_i}{\sqrt{\sum_{i\in P}|\alpha_i|^2}}\right).(b_{j+1,i}\otimes\cdots\otimes b_{mi}) \qquad (\mathsf{proj})$$

where $p = \sum_{i\in P}\big(\frac{|\alpha_i|^2}{\sum_{i=1}^{n}|\alpha_i|^2}\big)$, $j \leq n$, $k \leq n$, $\forall i \leq n$, $\forall h \leq m$, $b_{ih} = |0\rangle$ or $b_{ih} = |1\rangle$, if an α_i is absent, it is taken as 1, $\sum_{i=1}^{n}[\alpha_i.](b_{1i}\otimes\cdots\otimes b_{mi})$ is in normal form (hence, $1 \leq n \leq 2^m$), and $P \subseteq \mathbb{N}^{\leq n}$, such that $\forall i \in P$, $\forall h \leq j$, $b_{hi} = b_{hk}$.

Figure 4 Rewrite rules

casted to a term of type $S(B)$. The only valid casts are $S(S(A) \otimes B)$ and $S(A \otimes S(B))$ into $S(A \otimes B)$. Rules $\Uparrow^\alpha$ and $\Uparrow^+$ allow for compositional reasoning. This yields the system of Figure 3.

The rewrite system is given in Figure 4. It includes all the rules from Figure 2 plus the rules for tensors: (head) and (tail) to deal with lists, and the typing casts rules, which normalize superpositions to sums of base terms, while update the types.

The rule (proj) has been updated to account for multiple qubits systems. It normalizes (as in norm 1) the scalars on the obtained term. The call-by-base beta rule (β_b), and the contextual rule admitting reducing the argument on an application for the call-by-base abstraction are updated to allow for abstractions expecting arguments of type Q instead of just $\mathbb{Q}$ (that is, any base qubit type).

The first three rules in the group typing casts ($\mathsf{dist}_\mathsf{r}^+$), ($\mathsf{dist}_\mathsf{r}^\alpha$) and ($\mathsf{dist}_\mathsf{r}^0$), and their analogous ($\mathsf{dist}_\mathsf{l}^+$), ($\mathsf{dist}_\mathsf{l}^\alpha$) and ($\mathsf{dist}_\mathsf{l}^0$), deal with the distributivity of sums, scalar product and null vector respectively. If we ignore the type cast $\Uparrow_{S(A)}^{S(B)}$ on each rule, these rules are just distributivity rules. For example, rule ($\mathsf{dist}_\mathsf{r}^+$) acts on the term $(r+s)\otimes u$, distributing the sum with respect to the tensor product, producing $(r\otimes u + s\otimes u)$ (distribution to the right). However, the term $(r+s)\otimes u$ may have type $S(A)\otimes B$, $S(A)\otimes S(B)$ or $S(A\otimes B)$, while, among those, the term $(r\otimes u + s\otimes u)$ can only have type $S(A\otimes B)$. Hence, we cannot reduce the first term to the second without loosing subject reduction. Instead, we can mark the term in the following way: $\Uparrow_{S(S(A)\otimes B)}^{S(A\otimes B)} (r+s)\otimes u$, and hence this term will be typed only by $S(A\otimes B)$. Therefore, the distributivity rules will only work when the explicit cast indicates that the distribution can be done. Analogously, rule ($\mathsf{dist}_\mathsf{r}^\alpha$) rewrites $(\alpha.r)\otimes u$ into $\alpha.r\otimes u$ and rule ($\mathsf{dist}_\mathsf{r}^0$) rewrites $\vec{0}_{S(A)}\otimes u$ into $\vec{0}_{S(A\otimes B)}$.

Notice that in the previous example it would have been enough to use $\Uparrow_{S(A)\otimes B}^{S(A\otimes B)}$. Indeed, the term $(r+s)\otimes u$ can be typed with $S(A)\otimes B$. However, we prefer the more general $S(S(A)\otimes B)$ and hence to use the same rule when, for example, a sum is given.

The next two rules, ($\mathsf{dist}_\Uparrow^+$) and ($\mathsf{dist}_\Uparrow^\alpha$), distribute the cast over sums and scalars. For example

$$\begin{array}{l}\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})} ((\alpha.\left|1\right\rangle)\otimes\left|0\right\rangle + (\beta.\left|0\right\rangle)\otimes\left|1\right\rangle)\\ \xrightarrow{(\mathsf{dist}_\Uparrow^+)}_{(1)} (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})} (\alpha.\left|1\right\rangle)\otimes\left|0\right\rangle + \Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})} (\beta.\left|0\right\rangle)\otimes\left|1\right\rangle)\end{array}$$

and hence, the distributivity rule can act.

The last two rules in the group, ($\mathsf{neut}_\mathsf{r}^\Uparrow$) and ($\mathsf{neut}_\mathsf{l}^\Uparrow$), remove the cast when it is not needed anymore. For example

$$\begin{array}{rl}\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})} (\alpha.\beta.\left|0\right\rangle)\otimes\left|1\right\rangle \xrightarrow{(\mathsf{dist}_\mathsf{r}^\alpha)}_{(1)} & \alpha.\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})} (\beta.\left|0\right\rangle)\otimes\left|1\right\rangle\\ \xrightarrow{(\mathsf{dist}_\mathsf{r}^\alpha)}_{(1)} & \alpha.\beta.\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})} \left|0\right\rangle\otimes\left|1\right\rangle\\ \xrightarrow{(\mathsf{neut}_\mathsf{r}^\Uparrow)}_{(1)} & \alpha.\beta.\left|0\right\rangle\otimes\left|1\right\rangle\end{array}$$

The measurement rule (proj) is updated to measure the first j qubits. Hence, a n-qubits in normal form (that is, a sum of tensors of qubits with or without a scalar in front), for example, the term

$$((2.(\left|0\right\rangle\otimes\left|1\right\rangle\otimes\left|1\right\rangle) + \left|0\right\rangle\otimes\left|1\right\rangle\otimes\left|0\right\rangle) + 3.(\left|1\right\rangle\otimes\left|1\right\rangle\otimes\left|1\right\rangle))$$

can be measured and will produce a n-qubits where the first j qubits are the same and the remaining are untouched, with its scalars changed to have norm 1. In this 3-qubits example, measuring the first two can produce either $\left|0\right\rangle\otimes\left|1\right\rangle\otimes(\frac{2}{\sqrt{5}}.\left|1\right\rangle + \frac{1}{\sqrt{5}}.\left|0\right\rangle)$ or $\left|1\right\rangle\otimes\left|1\right\rangle\otimes(1.\left|1\right\rangle)$. The probability of producing the first is $\frac{|2|^2}{(|2|^2+|1|^2+|3|^2)} + \frac{|1|^2}{(|2|^2+|1|^2+|3|^2)} = \frac{5}{14}$ and the probability of producing the second is $\frac{|3|^2}{(|2|^2+|1|^2+|3|^2)} = \frac{9}{14}$.

Remark, to conclude, that because the calculus presented in this paper is call-by-base for the functions expecting a non-linear argument, it avoids a well-known problem in others λ-calculi with a linear logic type system including modalities. To illustrate this problem, consider the following typing judgement

$$y : S(\mathbb{Q}) \vdash (\lambda x : \mathbb{Q}\ (x \otimes x))(\pi y) : S(\mathbb{Q}) \otimes S(\mathbb{Q})$$

If we allow to β-reduce this term, we would obtain $(\pi y) \otimes (\pi y)$ which is not typable in the context $y : S(\mathbb{Q})$. A standard solution to this problem is illustrated in [7], where the terms that can be cloned are distinguished by a mark, and used in a *let* construction, while non-clonable terms are used in λ abstractions.

4 Subject reduction

Thanks to the explicit casts, the resulting system has the Subject Reduction property (Theorem 8), that is, the typing is preserved by weak-reduction (i.e. reduction on closed terms). This is the main theorem of this paper, and its proof is not trivial, specially due to the complexity of the system itself. The detailed proofs with some additional lemmas are given in the seven-page long Appendix B.

The two main lemmas of the proof, the generation lemma and the substitution lemma, are stated below, together with a few paradigmatic cases of the proof.

▶ **Definition 4.** We denote by $|\Gamma|$ to the set of types in Γ. For example, $|x : \mathbb{Q}, y : S(\mathbb{Q})| = \{\mathbb{Q}, S(\mathbb{Q})\}$.

▶ **Lemma 5** (Generation lemmas).

- *If* $\Gamma \vdash x : A$, *then* $x : \Psi \in \Gamma$, $|\Gamma| \setminus \{\Psi\} \subseteq \mathcal{B}$, *and* $\Psi \preceq A$.
- *If* $\Gamma \vdash \lambda x : \Psi\ t : A$, *then* $\Gamma', x : \Psi \vdash t : B$, *with* $\Gamma' \subseteq \Gamma$, $\Psi \Rightarrow B \preceq A$ *and* $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash |0\rangle : A$, *then* $\mathbb{Q} \preceq A$ *and* $|\Gamma| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash |1\rangle : A$, *then* $\mathbb{Q} \preceq A$ *and* $|\Gamma| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash \vec{0}_{S(B)} : A$, *then* $S(B) \preceq A$ *and* $|\Gamma| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash tu : A$, *then one of the following possibilities happens:*
  - $\Gamma_1 \vdash t : \Psi \Rightarrow B$ *and* $\Gamma_2 \vdash u : \Psi$, *with* $B \preceq A$, *or*
  - $\Gamma_1 \vdash t : S(\Psi \Rightarrow B)$ *and* $\Gamma_2 \vdash u : S(\Psi)$, *with* $S(B) \preceq A$.

  In both cases, $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$ *and* $|\Gamma| \setminus |\Gamma_1 \cup \Gamma_2| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash (t + u) : A$, *then* $\Gamma_1 \vdash t : B$ *and* $\Gamma_2 \vdash u : B$, *with* $S(B) \preceq A$ *and* $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$, $|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash \pi_j t : A$, *then* $\Gamma' \vdash t : Q_n^S$, *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $Q_n^{S\setminus\{1,\dots,j\}} \preceq A$.
- *If* $\Gamma \vdash ?\cdot : A$, *then* $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq A$ *and* $|\Gamma| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash \alpha.t : A$, *then* $\Gamma' \vdash t : B$, *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $S(B) \preceq A$.
- *If* $\Gamma \vdash t \otimes u : A$, *then* $\Gamma_1 \vdash t : B$ *and* $\Gamma_2 \vdash u : C$, *with* $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$, $|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| \subseteq \mathcal{B}$ *and* $B \otimes C \preceq A$.
- *If* $\Gamma \vdash \textit{head}\ t : A$, *then* $\Gamma' \vdash t : \mathbb{Q} \otimes Q$, *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $Q \preceq A$.
- *If* $\Gamma \vdash \textit{tail}\ t : A$, *then* $\Gamma' \vdash t : \mathbb{Q} \otimes Q$, *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $Q \preceq A$.
- *If* $\Gamma \vdash \Uparrow_{S(B)}^{S(C)} t : A$, *then* $\Gamma' \vdash t : S(B)$, *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $S(C) \preceq A$. *Moreover,*
  - *If* $t \neq \alpha.t'$ *and* $t \neq (t_1 + t_2)$, *then a rule* R *allows typing* $\Gamma' \vdash \Uparrow_{S(B)}^{S(C)} t : S(C)$ *directly from* $\Gamma' \vdash t : S(B)$.

- *If $t = \alpha.t'$, then $\Gamma' \vdash t' : S(B)$, and a rule R allows typing $\Gamma' \vdash\Uparrow^{S(C)}_{S(B)} t' : S(C)$ directly from $\Gamma' \vdash t' : S(B)$.*
- *If $t = (t_1 + t_2)$, then for $i = 1, 2$, $\Delta_i \vdash t_i : S(B)$, with $\Gamma' = \Delta_1, \Delta_2$, and a rule R allows typing $\Delta_i \vdash\Uparrow^{S(C)}_{S(B)} t_i : S(C)$ directly from $\Delta_i \vdash t_i : S(B)$.*

▶ **Corollary 6.** *If $\vdash (t + u) : A$, then $\vdash t : A$ and $\vdash u : A$.*

▶ **Lemma 7** (Substitution lemma)**.** *Let $FV(u) = \emptyset$, then if $\Gamma, x : \Psi \vdash t : A$, $\Delta \vdash u : \Psi$, where if $\Psi = Q$ then $u \in \mathcal{B}$, we have $\Gamma, \Delta \vdash (u/x)t : A$.*

Since the strategy is weak, subject reduction is proven for closed terms.

▶ **Theorem 8** (Subject reduction on closed terms)**.** *For any closed terms t and u and type A, if $t \longrightarrow_{(p)} u$ and $\vdash t : A$, then $\vdash u : A$.*

Proof. We only give two cases as example.

(β_b) and (β_n) Let $\vdash (\lambda x : \Psi\ t)u : A$, with $\vdash u : \Psi$, where, if $\Psi = Q$, then $u \in \mathcal{B}$. Then by Lemma 5, one of the following possibilities happens:
1. $\vdash \lambda x : \Psi\ t : \Psi' \Rightarrow B$ and $\vdash u : \Psi'$, with $B \preceq A$, or
2. $\vdash \lambda x : \Psi\ t : S(\Psi' \Rightarrow B)$ and $\vdash u : S(\Psi')$, with $S(B) \preceq A$.
Thus, in any case, by Lemma 5 again, $x : \Psi \vdash t : C$, with, in case 1, $\Psi \Rightarrow C \preceq \Psi' \Rightarrow B$ and in case 2, $\Psi \Rightarrow C \preceq S(\Psi' \Rightarrow B)$. Hence, $\Psi = \Psi'$ and in the first case $C \preceq B \preceq A$, while in the second, $C \preceq B \preceq S(B) \preceq A$, so, in general $C \preceq A$. Since $\vdash u : \Psi$, where if $\Psi = Q$, then $u \in \mathcal{B}$, by Lemma 7, $\vdash (u/x)t : C$, and by rule $\preceq$, $\vdash (u/x)t : A$.

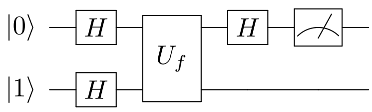
(proj) Let $\vdash \pi_j(\sum_{i=1}^n [\alpha_i.](b_{1i} \otimes \cdots \otimes b_{mi})) : A$. Then, by Lemma 5, we have that $\vdash \sum_{i=1}^n [\alpha_i.](b_{1i} \otimes \cdots \otimes b_{mi}) : Q^S$, with $Q^{S \setminus \{1,\dots,j\}} \preceq A$. Then, by Corollary 6, we have for every i, $\vdash b_{1i} \otimes \cdots \otimes b_{mi} : Q^S$. By Lemma 5, for every k, $\vdash b_{ki} : B_k$, such that $\bigotimes_k B_k \preceq Q^S$. Since b_{ki} are either $|0\rangle$ or $|1\rangle$, by Lemma 5, for every k, $\mathbb{Q} \preceq B_k$. Then, by rules $\preceq$, $\otimes_I$, α, S_I^+ and $\otimes_I$ again, we can derive $\vdash \bigotimes_{t=1}^j b_{tk} \otimes \sum_{i \in P} \left(\frac{\alpha_i}{\sqrt{\sum_{i \in P} |\alpha_i|^2}} \right) .(b_{j+1,i} \otimes \cdots \otimes b_{mi}) : Q^{S \setminus \{1,\dots,j\}}$, for any k and P such that $\forall i \in P$, $\forall t \leq j$, $b_{ti} = b_{tk}$. ◀

5 Examples

In this section we show that our language is expressive enough to express the Deutsch algorithm (Section 5.1) and the Teleportation algorithm (Section 5.2).

5.1 Deutsch algorithm

The Deutsch algorithm is given by the following circuit.



This algorithm tests whether the binary function f implemented by the oracle U_f is constant ($f(0) = f(1)$) or balanced ($f(0) \neq f(1)$). When the function is constant, the first qubit ends in $|0\rangle$, when it is balanced, it ends in $|1\rangle$.

The Hadamard gate (H) produces $\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle)$ when applied to $|0\rangle$ and $\frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle)$ when applied to $|1\rangle$. Hence, it can be implemented with the if-then-else construction:

$$\mathsf{H} = \lambda x : \mathbb{Q}\ \frac{1}{\sqrt{2}}.(|0\rangle + (x?(-|1\rangle)\cdot|1\rangle))$$

Notice that the abstracted variable has a base type (i.e. non-linear). Hence, if H is applied to a superposition, say $(\alpha.\,|0\rangle + \beta.\,|1\rangle)$, it reduces, as expected, in the following way:

$$\mathsf{H}(\alpha.\,|0\rangle + \beta.\,|1\rangle) \xrightarrow{(\mathsf{lin}_r^+)}_{(1)} (\mathsf{H}\alpha.\,|0\rangle + \mathsf{H}\beta.\,|1\rangle) \xrightarrow{(\mathsf{lin}_r^\alpha)^2}_{(1)} (\alpha.\mathsf{H}\,|0\rangle + \beta.\mathsf{H}\,|1\rangle)$$

and then is applied to the base terms.

We define H_1 as the function taking a two-qubits system and applying H to the first.

$$\mathsf{H}_1 = \lambda x : \mathbb{Q}\otimes\mathbb{Q}\ ((\mathsf{H}\ (\mathit{head}\ x)) \otimes (\mathit{tail}\ x))$$

Similarly, H_{both} applies H to both qubits.

$$\mathsf{H}_{both} = \lambda x : \mathbb{Q}\otimes\mathbb{Q}\ ((\mathsf{H}\ (\mathit{head}\ x)) \otimes (\mathsf{H}\ (\mathit{tail}\ x)))$$

The gate U_f is called *oracle*, and it is defined by $U_f\,|xy\rangle = |x, y \oplus f(x)\rangle$ where $\oplus$ is the addition modulo 2. In order to implement it, we need the *not* gate, which can be implemented similarly to the Hadamard gate:

$$\mathsf{not} = \lambda x : \mathbb{Q}\ (x?|0\rangle{\cdot}|1\rangle)$$

. Then, the U_f gate is implemented by:

$$\mathsf{U}_f = \lambda x : \mathbb{Q}\otimes\mathbb{Q}\ ((\mathit{head}\ x) \otimes ((\mathit{tail}\ x)?(\mathsf{not}\ (f\ (\mathit{head}\ x))){\cdot}(f\ (\mathit{head}\ x))))$$

where f is a given term of type $\mathbb{Q} \Rightarrow \mathbb{Q}$.

Finally, the Deutsch algorithm combines all the previous definitions:

$$\mathsf{Deutsch}_f = \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})} \mathsf{H}_1\ (\mathsf{U}_f\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\Uparrow^{S(\mathbb{Q}\otimes S(\mathbb{Q}))}_{S(S(\mathbb{Q})\otimes S(\mathbb{Q}))} \mathsf{H}_{both}\ (|0\rangle \otimes |1\rangle)))$$

The casts after the Hadamards are needed to fully develop the qubits and then being able to use it as an argument of a non-linear abstraction (i.e. an abstraction expecting for base terms and so linear-distributing over superpositions). The $\mathsf{Deutsch}_f$ term is typed, as expected, by

$$\vdash \mathsf{Deutsch}_f : \mathbb{Q} \otimes S(\mathbb{Q})$$

This term, on the identity function, reduces as follows:

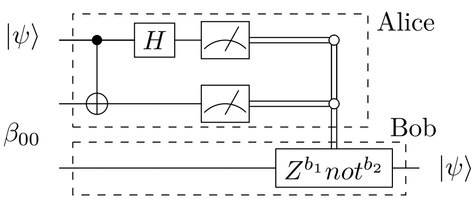
$$\mathsf{Deutsch}_{id} \longrightarrow^*_{(1)} \pi_1(\frac{1}{\sqrt{2}}.\,|1\rangle \otimes |0\rangle - \frac{1}{\sqrt{2}}.\,|1\rangle \otimes |1\rangle) \xrightarrow{(\mathsf{proj})}_{(1)} |1\rangle \otimes (\frac{1}{\sqrt{2}}.\,|0\rangle - \frac{1}{\sqrt{2}}.\,|1\rangle)$$

The trace on this reduction and the type derivation are given in Appendix C.

5.2 Teleportation algorithm

In the previous example the application of the measurement has only one possible outcome: The first qubit was already in a base state before measuring. Therefore, we introduce a sightly more complex example, the teleportation algorithm, where the measurement is used as an operator changing the state.

The circuit for this algorithm is the following:



The *cnot* gate can be implemented with an if-then-else construction as follows:

$$\mathsf{cnot} = \lambda x : \mathbb{Q} \otimes \mathbb{Q}\ ((\mathit{head}\ x) \otimes ((\mathit{head}\ x)?(\mathsf{not}\ (\mathit{tail}\ x))\cdot(\mathit{tail}\ x)))$$

We define H_1^3 to apply H to the first qubit of a three-qubit system.

$$\mathsf{H}_1^3 = \lambda x : \mathbb{Q} \otimes \mathbb{Q} \otimes \mathbb{Q}\ ((\mathsf{H}\ (\mathit{head}\ x)) \otimes (\mathit{tail}\ x))$$

Remark that the only difference with H_1 is the type of the abstracted variable. In addition, we need to apply *cnot* to the two first qubits, so we define cnot_{12}^3 as

$$\mathsf{cnot}_{12}^3 = \lambda x : \mathbb{Q} \otimes \mathbb{Q} \otimes \mathbb{Q}\ ((\mathsf{cnot}\ (\mathit{head}\ x \otimes (\mathit{head}\ \mathit{tail}\ x))) \otimes (\mathit{tail}\ \mathit{tail}\ x))$$

The Z gate returns $|0\rangle$ when it receives $|0\rangle$, and $-|1\rangle$ when it receives $|1\rangle$. Hence, it can be implemented by:

$$\mathsf{Z} = \lambda x : \mathbb{Q}\ (x?(-|1\rangle)\cdot|0\rangle)$$

The Bob side of the algorithm will apply Z and/or *not* according to the bits it receives from Alice. Hence, for any $\vdash \mathsf{U} : \mathbb{Q} \Rightarrow S(\mathbb{Q})$ or $\vdash \mathsf{U} : \mathbb{Q} \Rightarrow \mathbb{Q}$, we define $\mathsf{U}^{(b)}$ to be the function which depending on the value of a base qubit b applies the U gate or not:

$$\mathsf{U}^{(b)} = (\lambda x : \mathbb{Q}\ \lambda y : \mathbb{Q}\ (x?\mathsf{U}y\cdot y))\ b$$

Alice and Bob parts of the algorithm are defined separately:

$$\mathsf{Alice} = \lambda x : S(\mathbb{Q}) \otimes S(\mathbb{Q} \otimes \mathbb{Q})\ (\pi_2(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1^3\ (\mathsf{cnot}_{12}^3 \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}\otimes\mathbb{Q}))} \Uparrow^{S(\mathbb{Q}\otimes S(\mathbb{Q}\otimes\mathbb{Q}))}_{S(S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q}))}\ x)))$$

Notice that before passing to cnot_{12}^3 the parameter of type $S(\mathbb{Q}) \otimes S(\mathbb{Q} \otimes \mathbb{Q})$, we need to fully develop the term using the two casts, and again, after the Hadamard gate. Bob side is implemented by

$$\mathsf{Bob} = \lambda x : \mathbb{Q} \otimes \mathbb{Q} \otimes \mathbb{Q}\ (\mathsf{Z}^{(\mathit{head}\ x)}(\mathsf{not}^{(\mathit{head}\ \mathit{tail}\ x)}\ (\mathit{tail}\ \mathit{tail}\ x)))$$

The teleportation is applied to an arbitrary qubit and to the following Bell state $\beta_{00} = (\frac{1}{\sqrt{2}}.|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}.|1\rangle \otimes |1\rangle)$ and it is defined by:

$$\mathsf{Teleportation} = \lambda q : S(\mathbb{Q})\ (\mathsf{Bob}(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q}))}\ \mathsf{Alice}\ (q \otimes \beta_{00})))$$

This term is typed, as expected, by:

$$\vdash \mathsf{Teleportation} : S(\mathbb{Q}) \Rightarrow S(\mathbb{Q})$$

and applying the teleportation to any superposition $(\alpha.|0\rangle + \beta.|1\rangle)$ will reduce, as expected, to $(\alpha.|0\rangle + \beta.|1\rangle)$. The trace on this reduction and the type derivation are given in Appendix D.

6 Conclusion

In this paper we have proposed a way to unify logic-linear and algebraic-linear quantum λ-calculi, by interpreting λ-terms as linear functions when they expect duplicable data and as non-linear ones when they do not, and illustrated this idea with the definition of a calculus.

This calculus is first-order in the sense that variables do not have functional types. In a higher-order we should expect abstractions to be clonable. But, allowing cloning abstractions allows cloning superpositions, by hiding them inside. For example, $\lambda x : \mathbb{Q} \Rightarrow$

$\mathbb{Q}\ (\frac{1}{\sqrt{2}}.\,|0\rangle + \frac{1}{\sqrt{2}}.\,|1\rangle)$. We could argue [4, 5] that what is cloned is not the superposition but a function that creates the superposition, because we have no way there to create such an abstraction from an arbitrary superposition. The situation is different in the calculus presented in this paper as the term $\lambda x : S(\mathbb{Q})\ \lambda y : \mathbb{Q}\ x$ precisely takes any term t of type $S(\mathbb{Q})$ and returns the term $\lambda y : \mathbb{Q}\ t$. So, a cloning machine could be constructed. Extending this calculus to the higher-order will require characterizing precisely the abstractions that can be taken as arguments.

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A Basics notions of quantum computing

This appendix does not pretend to introduce a full description of quantum computing, the interested reader can find actual introductions to this area in many textbooks, e.g. [11, 12]. This section only pretends to introduce some basic notations and concepts.

In quantum computation, data is expressed by normalised vectors in Hilbert spaces. For our purpose, this means that the vector spaces are defined over complex numbers and come with a norm and a notion of orthogonality. The smallest space usually considered is the space of *qubits*. This space is the two-dimensional vector space $\mathbb{C}^2$, and it comes with a chosen orthonormal basis denoted by $\{|0\rangle, |1\rangle\}$. A qubit (or quantum bit) is a normalised vector $\alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. To denote an unknown qubit ψ it is common to write $|\psi\rangle$. A two-qubits vector is a normalised vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$, that is, a normalised vector generated by the orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where $|xy\rangle$ stands for $|x\rangle \otimes |y\rangle$. In the same way, a n-qubits vector is a normalised vector in $(\mathbb{C}^2)^n$ (or $\mathbb{C}^N$ with $N = 2^n$). Also common is the notation $\langle\psi|$ for the transposed, conjugate of $|\psi\rangle$, e.g. if $|\psi\rangle = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$, then $\langle\psi| = [\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*]$ where for any $\alpha \in \mathbb{C}$, α^* denotes its conjugate.

The operators on qubits that are considered in this paper are the *quantum gates*, that is, isometric operators. An *isometric operator* is a linear function preserving the norm and the orthogonality of vectors. The *adjoint* of a given operator U is denoted by $U^\dagger$, and the isometric condition imposes that $U^\dagger U = Id$. These functions are linear, and so it is enough to describe their action on the base vectors. Another way to describe these functions would be by matrices, and then the adjoint is just its conjugate transpose. A set of universal quantum gates is the set *cnot*, $R_{\frac{\pi}{4}}$ and *had*, which can be defined as follows:

The *cnot* gate. The *controlled-not* is a two-qubits gate which only changes the second qubit if the first one is $|1\rangle$:
$\mathit{cnot}\,|0x\rangle = |0x\rangle \quad ; \quad \mathit{cnot}\,|1x\rangle = |1\rangle \otimes \mathit{not}\,|x\rangle$ where $\mathit{not}\,|0\rangle = |1\rangle$ and $\mathit{not}\,|1\rangle = |0\rangle$.

The $R_{\frac{\pi}{4}}$ gate. The $R_{\frac{\pi}{4}}$ gate is a single-qubit gate that modifies the *phase* of the qubit:
$R_{\frac{\pi}{4}}|0\rangle = |0\rangle \quad ; \quad R_{\frac{\pi}{4}}|1\rangle = e^{i\frac{\pi}{4}}|1\rangle$ where $\frac{\pi}{4}$ is the phase shift.

The H gate. The *Hadamard* gate is a single-qubit gate which produces a basis change:
$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad ; \quad H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

To make these gates act in higher-dimension qubits, they can be put together with the bilinear symbol $\otimes$. For example, to make the Hadamard gate act only in the first qubit of a two-qubits register, it can be taken to $H \otimes Id$, and to apply a Hadamard gate to both qubits, just $H \otimes H$.

An important restriction, which has to be taken into account if a calculus pretends to encode quantum computing, is the so called *no-cloning theorem* [16]:

▶ **Theorem 9** (No cloning). *There is no linear operator such that, given any qubit $|\phi\rangle \in \mathbb{C}^N$, it can clone it. That is, it does not exist any isometric operator U and fixed $|\psi\rangle \in \mathbb{C}^N$ such that $U|\psi\phi\rangle = |\phi\phi\rangle$.*

Proof. Assume there exists such an operator U, so given any $|\varphi\rangle$ and $|\phi\rangle$ one has $U|\psi\varphi\rangle = |\varphi\varphi\rangle$ and also $U|\psi\phi\rangle = |\phi\phi\rangle$. Then

$$\langle U\varphi\psi|U\psi\phi\rangle = \langle\varphi\varphi|\phi\phi\rangle \tag{1}$$

where $\langle U\varphi\psi|$ is the conjugate transpose of $U|\psi\varphi\rangle$. However, notice that the left side of equation (1) can be rewritten as

$$\langle\varphi\psi|U^\dagger U|\psi\phi\rangle = \langle\varphi\psi|\psi\phi\rangle = \langle\varphi|\phi\rangle$$

While the right side of equation (1) can be rewritten as

$$\langle\varphi|\phi\rangle\langle\varphi|\phi\rangle = \langle\varphi|\phi\rangle^2$$

So $\langle\varphi|\phi\rangle = \langle\varphi|\phi\rangle^2$, which implies either $\langle\varphi|\phi\rangle = 0$ or $\langle\varphi|\phi\rangle = 1$, none of which can be assumed in the general case, since $|\varphi\rangle$ and $|\phi\rangle$ were picked as random qubits. ◀

The implication of this theorem in the design choices of a calculus is that it must be forbidden to allow functions duplicating arbitrary arguments. However, notice that this does not forbid cloning some specific qubit states. Indeed, for example the qubits $|0\rangle$ and $|1\rangle$ can be cloned without much effort by using the *cnot* gate: $\mathit{cnot}\,|00\rangle = |00\rangle$ and $\mathit{cnot}\,|10\rangle = |11\rangle$. In this sense, the imposed restriction is not a resources-aware restriction *à la* linear logic [9]. It is a restriction that forbids us from creating a 'universal cloning machine', but still allows us to clone any given known term.

Another operation considered on qubits is the measurement. A projector is an operator of the form $|\phi\rangle\langle\phi|$. For example, in the canonical base $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$, $P_0 = |0\rangle\langle 0|$ is a projector and $P_1 = |1\rangle\langle 1|$ is another projector, with respect to such a base. Indeed,

$$\begin{aligned}
P_0(\alpha|0\rangle + \beta|1\rangle) &= \alpha P_0|0\rangle + \beta P_0|1\rangle \\
&= \alpha|0\rangle\langle 0|0\rangle + \beta|0\rangle\langle 0|1\rangle \\
&= \alpha|0\rangle \\
P_1(\alpha|0\rangle + \beta|1\rangle) &= \alpha P_1|0\rangle + \beta P_1|1\rangle \\
&= \alpha|1\rangle\langle 1|0\rangle + \beta|1\rangle\langle 1|1\rangle \\
&= \beta|1\rangle
\end{aligned}$$

With these projectors we can define the measurement operators M_0 and M_1 as

$$M_i|\psi\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$$

For example,

$$\begin{aligned}
M_0(\alpha|0\rangle + \beta|1\rangle) &= \frac{P_0(\alpha|0\rangle + \beta|1\rangle)}{\sqrt{(\alpha^*\langle 0| + \beta^*\langle 1|)P_0(\alpha|0\rangle + \beta|1\rangle)}} \\
&= \frac{\alpha|0\rangle}{\sqrt{(\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle)}} \\
&= \frac{\alpha|0\rangle}{\sqrt{|\alpha|^2\langle 0|0\rangle + \beta^*\alpha\langle 1|0\rangle}} \\
&= \frac{\alpha|0\rangle}{\sqrt{|\alpha|^2}} = \frac{\alpha}{|\alpha|}|0\rangle \ \equiv^2 \ |0\rangle
\end{aligned}$$

The quantum measurement is defined in terms of sets of measurements operators. For example, in the canonical base $\{|0\rangle, |1\rangle\}$, the set $\{M_0, M_1\}$ is a quantum measurement. When it acts on a qubit $|\phi\rangle$, it will apply the operator M_i, with probability $\langle\psi|P_i|\psi\rangle$.

2 The scalar $\frac{\alpha}{|\alpha|}$ is known as a *phase* and can be ignored, so only $|0\rangle$ remains.

B Omitted proofs in Section 4 (Subject reduction)

▶ **Lemma 10.** *If $S(A) \preceq B$, then there exists C such that $B = S(C)$*

Proof. Straightforward analysis of the definition of $\preceq$. ◀

Lemma 5 (Generation lemmas).

- *If $\Gamma \vdash x : A$, then $x : \Psi \in \Gamma$, $|\Gamma| \setminus \{\Psi\} \subseteq \mathcal{B}$, and $\Psi \preceq A$.*
- *If $\Gamma \vdash \lambda x : \Psi\ t : A$, then $\Gamma', x : \Psi \vdash t : B$, with $\Gamma' \subseteq \Gamma$, $\Psi \Rightarrow B \preceq A$ and $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$.*
- *If $\Gamma \vdash |0\rangle : A$, then $\mathbb{Q} \preceq A$ and $|\Gamma| \subseteq \mathcal{B}$.*
- *If $\Gamma \vdash |1\rangle : A$, then $\mathbb{Q} \preceq A$ and $|\Gamma| \subseteq \mathcal{B}$.*
- *If $\Gamma \vdash \vec{0}_{S(B)} : A$, then $S(B) \preceq A$ and $|\Gamma| \subseteq \mathcal{B}$.*
- *If $\Gamma \vdash tu : A$, then one of the following possibilities happens:*
  - *$\Gamma_1 \vdash t : \Psi \Rightarrow B$ and $\Gamma_2 \vdash u : \Psi$, with $B \preceq A$, or*
  - *$\Gamma_1 \vdash t : S(\Psi \Rightarrow B)$ and $\Gamma_2 \vdash u : S(\Psi)$, with $S(B) \preceq A$.*

  In both cases, $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$ and $|\Gamma| \setminus |\Gamma_1 \cup \Gamma_2| \subseteq \mathcal{B}$.
- *If $\Gamma \vdash (t + u) : A$, then $\Gamma_1 \vdash t : B$ and $\Gamma_2 \vdash u : B$, with $S(B) \preceq A$ and $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$, $|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| \subseteq \mathcal{B}$.*
- *If $\Gamma \vdash \pi_j t : A$, then $\Gamma' \vdash t : Q_n^S$, with $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ and $Q_n^{S\setminus\{1,\dots,j\}} \preceq A$.*
- *If $\Gamma \vdash ?\cdot : A$, then $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq A$ and $|\Gamma| \subseteq \mathcal{B}$.*
- *If $\Gamma \vdash \alpha.t : A$, then $\Gamma' \vdash t : B$, with $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ and $S(B) \preceq A$.*
- *If $\Gamma \vdash t \otimes u : A$, then $\Gamma_1 \vdash t : B$ and $\Gamma_2 \vdash u : C$, with $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$, $|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| \subseteq \mathcal{B}$ and $B \otimes C \preceq A$.*
- *If $\Gamma \vdash \mathit{head}\ t : A$, then $\Gamma' \vdash t : \mathbb{Q} \otimes Q$, with $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ and $Q \preceq A$.*
- *If $\Gamma \vdash \mathit{tail}\ t : A$, then $\Gamma' \vdash t : \mathbb{Q} \otimes Q$, with $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ and $Q \preceq A$.*
- *If $\Gamma \vdash \Uparrow_{S(B)}^{S(C)} t : A$, then $\Gamma' \vdash t : S(B)$, with $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ and $S(C) \preceq A$. Moreover,*
  - *If $t \neq \alpha.t'$ and $t \neq (t_1 + t_2)$, then a rule R allows typing $\Gamma' \vdash \Uparrow_{S(B)}^{S(C)} t : S(C)$ directly from $\Gamma' \vdash t : S(B)$.*
  - *If $t = \alpha.t'$, then $\Gamma' \vdash t' : S(B)$, and a rule R allows typing $\Gamma' \vdash \Uparrow_{S(B)}^{S(C)} t' : S(C)$ directly from $\Gamma' \vdash t' : S(B)$.*
  - *If $t = (t_1 + t_2)$, then for $i = 1, 2$, $\Delta_i \vdash t_i : S(B)$, with $\Gamma' = \Delta_1, \Delta_2$, and a rule R allows typing $\Delta_i \vdash \Uparrow_{S(B)}^{S(C)} t_i : S(C)$ directly from $\Delta_i \vdash t_i : S(B)$.*

Proof. First notice that if $\Gamma \vdash t : A$ is derivable, then $\Delta \vdash t : B$ is derivable, with $\Gamma \subseteq \Delta$ and $|\Delta \setminus \Gamma| \subseteq \mathcal{B}$ (because of rule W) and $A \preceq B$, (because of rule $\preceq$). Notice also that those are the only typing rules changing the sequent without changing the term on the sequent. Rules $\Rightarrow_E$ and $\Rightarrow_{ES}$ and are straightforward to check. All the other rules, except for those involving arrows, are syntax directed: one rule for each term. Therefore, the lemma is proven by a straightforward rule by rule analysis for all the items but last.

For the last item, let $\Gamma \vdash \Uparrow_{S(B)}^{S(C)} t : A$, then it is clear that $\Gamma' \vdash \Uparrow_{S(B)}^{S(C)} t : S(C)$, with $S(C) \preceq A$, $\Gamma' \subseteq \Gamma$ and $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$. Also, notice that there is always a Rule $\Uparrow$ allowing typing $\vdash \Uparrow_{S(D)}^{S(E)} u : S(E)$ if $\vdash u : S(D)$, if $\Uparrow_{S(D)}^{S(E)}$ is a subterm of a well-typed term.

We proceed by induction on t.

- If $t \neq \alpha.t'$ and $t \neq (t_1 + t_2)$, then the only possibility is $\Gamma' \vdash t : S(B)$.
- If $t = \alpha.t'$, then the cases are the following:
  - $\Gamma' \vdash \alpha.t : S(B)$, then by a previous item on this Lemma, $\Gamma'' \vdash t' : C$, with $S(C) \preceq S(B)$, $\Gamma'' \subseteq \Gamma'$ and $|\Gamma' \setminus \Gamma''| \subseteq \mathcal{B}$. Hence, by rules W and $\preceq$, $\Gamma' \vdash t' : S(B)$.
  - $\Gamma' \vdash \Uparrow_{S(B)}^{S(C)} t' : S(C)$. Then, by the induction hypothesis, $\Gamma'' \vdash t' : S(B)$, with $\Gamma'' \subseteq \Gamma'$ and $|\Gamma' \setminus \Gamma''| \subseteq \mathcal{B}$. Hence, by rule W $\Gamma' \vdash t' : S(B)$.
- If $t = (t_1 + t_2)$, then the cases are the following:

- $\Gamma' \vdash (t_1+t_2) : S(B)$, then by a previous item in this Lemma, for $i=1,2$ $\Delta_i \vdash t_i : S(C)$, with $S(C) \preceq S(B)$, $\Delta_1' \cup \Delta_2' \subseteq \Gamma'$ and $|\Gamma' \setminus (\Delta_1' \cup \Delta_2')| \subseteq \mathcal{B}$. Hence, by rules W and $\preceq$, $\Delta_i \vdash t_i : S(B)$.
- $\Delta_i \vdash\Uparrow_{S(B)}^{S(C)} t_i : S(C)$, for $i=1,2$. Then, by the induction hypothesis, $\Delta_i' \vdash t_i : S(B)$, with $\Delta_i' \subseteq \Delta_i$ and $|\Delta_i \setminus \Delta_i'| \subseteq \mathcal{B}$. Hence, by rule W $\Delta_i \vdash t_i : S(B)$. ◀

Corollary 6.
- *If* $\vdash (t+u) : A$, *then* $\vdash t : A$ *and* $\vdash u : A$.
- *If* $\vdash (t+u) : A$, *then* $A = S(B)$.
- *If* $\vdash \alpha.t : A$, *then* $\vdash t : A$.
- *If* $\vdash \alpha.t : A$, *then* $\vdash \beta.t : A$.
- *If* $\vdash \alpha.t : A$, *then* $A = S(B)$.

Proof.
- By Lemma 5, $\vdash t : B$ and $\vdash u : B$, with $B \preceq S(B) \preceq A$, then, we conclude by rule $\preceq$.
- By Lemma 5, $\vdash t : C$ and $\vdash u : C$, with $C \preceq S(C) \preceq A$, but then, by Lemma 10, $A = S(B)$ for some type B.
- By Lemma 5, $\vdash t : B$, with $S(B) \preceq A$, then, we conclude by rule $\preceq$.
- By Lemma 5, $\vdash t : B$, with $S(B) \preceq A$, then we conclude by rules S_I^α and $\preceq$.
- By Lemma 5, $\vdash t : C$ with $S(C) \preceq A$, but then, by Lemma 10, $A = S(B)$ for some type B. ◀

▶ **Corollary 11.** *If* $b \in \mathcal{B}$ *and* $\vdash b : S(A)$, *then* $\vdash b : A$.

Proof. We proceed by induction on b.
- Let $b = \lambda x : \Psi\ t$. Then, by Lemma 5, $x : \Psi \vdash t : B$, with $\Psi \Rightarrow B \preceq S(A)$, and so $\Psi \Rightarrow B \preceq A$, and we conclude by rule $\preceq$.
- Let $b = |0\rangle$. Then, by Lemma 5, $\mathbb{Q} \preceq S(A)$, hence $\mathbb{Q} \preceq A$ and we conclude by rule $\preceq$.
- Let $b = |1\rangle$. Analogous to previous case.
- Let $b = b_1 \otimes b_2$. Then, by Lemma 5, $\vdash b_1 : B_1$, $\vdash b_2 : B_2$, and $B_1 \otimes B_2 \preceq S(A)$. Hence, $B_1 \otimes B_2 \preceq A$ and we conclude by rule $\preceq$. ◀

▶ **Lemma 12.** *If* $\Gamma \vdash t : A$ *and* $FV(t) = \emptyset$, *then* $|\Gamma| \subseteq \mathcal{B}$.

Proof. If $FV(t) = \emptyset$ then $\vdash t : A$. If $\Gamma \neq \emptyset$, the only way to derive $\Gamma \vdash t : A$ is by using rule W to form Γ, hence $|\Gamma| \subseteq \mathcal{B}$. ◀

Lemma 7 (Substitution lemma). *Let* $FV(u) = \emptyset$, *then if* $\Gamma, x : \Psi \vdash t : A$, $\Delta \vdash u : \Psi$, *where if* $\Psi = Q$ *then* $u \in \mathcal{B}$, *we have* $\Gamma, \Delta \vdash (u/x)t : A$.

Proof. Notice that due to Lemma 12, $|\Delta| \subseteq \mathcal{B}$, hence, it suffices to consider $\Delta = \emptyset$. We proceed by structural induction on t.

The set of terms be divided in the following groups:

$$\begin{aligned}
\mathsf{unclassified} &:= x \mid \lambda x : \Psi\ t\\
\mathsf{arity}^0 &:= \vec{0}_{S(A)} \mid |0\rangle \mid |1\rangle \mid ?\cdot\\
\mathsf{arity}^1(r) &:= \pi_j r \mid \alpha.r \mid \mathit{head}\ r \mid \mathit{tail}\ r \mid \Uparrow_{S(B)}^{S(C\otimes D)} t\\
\mathsf{arity}^2(r)(s) &:= rs \mid (r+s) \mid r \otimes s
\end{aligned}$$

Hence, we can consider the terms by groups:

unclassified terms

$t = x$. By Lemma 5, $A = \Psi$, $|\Gamma| \subseteq \mathcal{B}$ and $\Psi \preceq A$. Since $(u/x)x = u$, we have $\vdash (u/x)x : \Psi$. Hence, since $\Psi \preceq A$, by rule $\preceq$, $\vdash (u/x)x : A$. Finally, since $|\Gamma| \subseteq \mathcal{B}$, by rule W, we have $\Gamma \vdash (u/x)x : A$.

$t = y \neq x$. By Lemma 5, $y : \Psi' \in \Gamma$, $(|\Gamma| \cup \{\Psi\}) \setminus \{\Psi'\} \subseteq \mathcal{B}$ and $\Psi' \preceq A$. Hence, by rule $\preceq$, $y : \Psi' \vdash y : A$. Since $|\Gamma| \subseteq \mathcal{B}$, by rule W, we have $\Gamma \vdash y : A$. Finally, since $(u/x)y = y$, we have $\Gamma \vdash (u/x)y : A$.

$t = \lambda y : \Psi'\ v$. Without lost of generality, assume y is does not appear free in u. By Lemma 5, $\Gamma', y : \Psi' \vdash v : B$, with $\Gamma' \subseteq \Gamma \cup \{x : \Psi\}$, $\Psi' \Rightarrow B \preceq A$ and $(|\Gamma| \cup \{\Psi\}) \setminus |\Gamma'| \subseteq \mathcal{B}$. By the induction hypothesis, $\Gamma'', y : \Psi' \vdash (u/x)v : B$, with $\Gamma'' = \Gamma' \setminus \{x : \Psi\}$. Notice that if $x : \Psi \in \Gamma'$, the induction hypothesis applies directly, in other case, $\Psi \in \mathcal{B}$ and so by rule W the context can be enlarged to include $x : \Psi$, hence the induction hypothesis applies in any case. Therefore, by rule $\Rightarrow_I$, $\Gamma'' \vdash \lambda y : \Psi'\ (u/x)v : \Psi' \Rightarrow B$. Since $\Psi' \Rightarrow B \preceq A$, by rule $\preceq$, $\Gamma'' \vdash \lambda y : \Psi'\ (u/x)v : A$. Hence, since $|\Gamma| \setminus |\Gamma''| \subseteq \mathcal{B}$, by rule W, $\Gamma \vdash \lambda y : \Psi'\ (u/x)v : A$. Since y does not appear free in u, $\lambda y : \Psi'\ (u/x)v = (u/x)(\lambda y : \Psi'\ v)$. Therefore, $\Gamma \vdash (u/x)(\lambda y : \Psi'\ v) : A$.

arity^0 terms All of these terms are typed by an axiom with a type B which, by Lemma 5, $B \preceq A$. Also, by the same Lemma, $|\Gamma, x : \Psi| \subseteq \mathcal{B}$. So, we can type with the axiom, and empty context, $\vdash \mathsf{arity}^0 : B$, and so, by rule W, $\Gamma \vdash \mathsf{arity}^0 : B$. Notice that $\mathsf{arity}^0 = (u/x)\mathsf{arity}^0$. We conclude by rule $\preceq$.

$\mathsf{arity}^1(r)$ terms By Lemma 5, $\Gamma' \vdash r : B$, such that by a derivation tree T, $\Gamma' \vdash \mathsf{arity}^1(r) : C$, where $\Gamma' \subseteq (\Gamma \cup \{x : \Psi\})$, $(|\Gamma| \cup \Psi) \setminus |\Gamma'| \subseteq \mathcal{B}$ and $C \preceq A$. If $x : \Psi \notin \Gamma'$, then $\Psi = Q$ and so we can extend Γ' with $x : \Psi$. Hence, in any case, by the induction hypothesis, $\Gamma' \setminus \{x : \Psi\} \vdash (u/x)r : C$. Then, using the derivation tree T, $\Gamma' \setminus \{x : \Psi\} \vdash \mathsf{arity}^1((u/x)r) : C$. Notice that $\mathsf{arity}^1((u/x)r) = (u/x)\mathsf{arity}^1(r)$. We conclude by rules W and $\preceq$.

$\mathsf{arity}^2(r)(s)$ terms By Lemma 5, $\Gamma_1 \vdash r : C$ and $\Gamma_2 \vdash s : D$, such that by a typing rule R, $\Gamma_1, \Gamma_2 \vdash \mathsf{arity}^2(r)(s) : E$, with $E \preceq A$, and where $(\Gamma_1 \cup \Gamma_2) \subseteq (\Gamma \cup \{x : \Psi\})$ and $(|\Gamma| \cup \Psi) \setminus (|\Gamma_1| \cup |\Gamma_2|) \subseteq \mathcal{B}$. Therefore, if $x : \Psi \notin \Gamma_i$, $i = 1, 2$, we can extend Γ_i with $x : \Psi$ using rule W. Hence, by the induction hypothesis, $\Gamma_1 \setminus \{x : \Psi\} \vdash (u/x)r : C$ and $\Gamma_2 \setminus \{x : \Psi\} \vdash (u/x)s : D$. So, by rule R, $\Gamma_1 \setminus \{x : \Psi\}, \Gamma_2 \setminus \{x : \Psi\} \vdash \mathsf{arity}^2((u/x)r)((u/x)s) : E$. Notice that $\mathsf{arity}^2((u/x)r)((u/x)s) = (u/x)\mathsf{arity}^2(r)(s)$. We conclude by rules W and $\preceq$. ◀

Theorem 8 (Subject reduction on closed terms). *For any closed terms t and u and type A, if $t \longrightarrow_{(p)} u$ and $\vdash t : A$, then $\vdash u : A$.*

Proof. We proceed by induction on the rewrite relation.

(β_b) and (β_n) Let $\vdash (\lambda x : \Psi\ t)u : A$, with $\vdash u : \Psi$, where, if $\Psi = Q$, then $u \in \mathcal{B}$. Then by Lemma 5, one of the following possibilities happens:

1. $\vdash \lambda x : \Psi\ t : \Psi' \Rightarrow B$ and $\vdash u : \Psi'$, with $B \preceq A$, or
2. $\vdash \lambda x : \Psi\ t : S(\Psi' \Rightarrow B)$ and $\vdash u : S(\Psi')$, with $S(B) \preceq A$.

Thus, in any case, by Lemma 5 again, $x : \Psi \vdash t : C$, with, in case 1, $\Psi \Rightarrow C \preceq \Psi' \Rightarrow B$ and in case 2, $\Psi \Rightarrow C \preceq S(\Psi' \Rightarrow B)$. Hence, $\Psi = \Psi'$ and in the first case $C \preceq B \preceq A$, while in the second, $C \preceq B \preceq S(B) \preceq A$, so, in general $C \preceq A$. Since $\vdash u : \Psi$, where if $\Psi = Q$, then $u \in \mathcal{B}$, by Lemma 7, $\vdash (u/x)t : C$, and by rule $\preceq$, $\vdash (u/x)t : A$.

(neutral) Let $\vdash (\vec{0}_{S(A)} + t) : A$. Then, by Corollary 6, $\vdash t : A$.

(unit) Let $\vdash 1.t : A$. Then, by Corollary 6, $\vdash t : A$.

(zero$_\alpha$) Let $\vdash 0.t : A$. Then, by Corollary 6, $A = S(B)$, and so by rule $Ax_{\vec{0}}$, $\vdash \vec{0}_{S(A)} : A$.

(zero) Let $\vdash \alpha.0 : A$. By Corollary 6, $A = S(B)$, hence, by rule $Ax_{\vec{0}}$, $\vdash \vec{0}_{S(A)} : A$.

(prod) Let $\vdash \alpha.(\beta.t) : A$. By Corollary 6, $\vdash \beta.t : A$. Then, by Corollary 6 again, $\vdash (\alpha \times \beta).t : A$.

(αdist) Let $\vdash \alpha.(t + u) : A$. By Lemma 5, $\vdash (t + u) : B$, with $S(B) \preceq A$. Then, by Corollary 6, $\vdash t : B$ and $\vdash u : B$. Hence, by rule S_I^α, $\vdash \alpha.t : S(B)$ and $\vdash \alpha.u : S(B)$. We conclude by rules S_I^+ and $\preceq$.

(fact) Let $\vdash (\alpha.t + \beta.t) : A$. By Corollary 6, $\vdash \alpha.t : A$. Then, by Corollary 6 again, $\vdash (\alpha + \beta).t : A$.

(fact^1) Let $\vdash (\alpha.t + t) : A$. By Corollary 6, $\vdash \alpha.t : A$. Then, by Corollary 6 again, $\vdash (\alpha + 1).t : A$.

(fact^2) Let $\vdash (t + t) : A$. By Lemma 5, $\vdash t : B$, with $S(B) \preceq A$. Then, by rule S_I^α, $\vdash 2.t : S(B)$. We conclude by rule $\preceq$.

(head) Let $\vdash \mathit{head}\ (t \otimes u) : A$. Hence, by Lemma 5, $\vdash t \otimes u : Q \otimes Q$, with $Q \preceq A$. Then, by Lemma 5 again, $\vdash t : B$ and $\vdash u : C$, with $B \otimes C \preceq Q \otimes Q$. Hence, $B \preceq Q \preceq A$, and so we conclude by rule $\preceq$.

(tail) Analogous to case (head).

(if$_1$) Let $\vdash |1\rangle ?u{\cdot}v : A$. Then, by Lemma 5, one of the following possibilities happens:

1. $\vdash |1\rangle ?u{\cdot} : \Psi_1 \Rightarrow B$ and $\vdash v : \Psi_1$, with $B \preceq A$. Then, by Lemma 5 again, one of the following possibilities happens:
   a. $\vdash |1\rangle ?{\cdot} : \Psi_2 \Rightarrow C$, and $\vdash u : \Psi_2$, with $C \preceq \Psi_1 \Rightarrow B \preceq \Psi_1 \Rightarrow A$. Then, by Lemma 5 again, one of the following possibilities happens:
      i. $\vdash ?{\cdot} : \Psi_3 \Rightarrow D$, and $\vdash |1\rangle : \Psi_3$, with $D \preceq \Psi_2 \Rightarrow C \preceq \Psi_2 \Rightarrow \Psi_2 \Rightarrow A$. Then, by Lemma 5 again, $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq \Psi_3 \Rightarrow \Psi_2 \Rightarrow \Psi_1 \Rightarrow A$, so $\Psi_2 = \mathbb{Q} \preceq A$, and hence, by rule $\preceq$, $\vdash u : A$.
      ii. $\vdash ?{\cdot} : S(\Psi_3 \Rightarrow D)$ and $\vdash |1\rangle : S(\Psi_3)$, with $S(D) \preceq \Psi_2 \Rightarrow C$, but this case is impossible by Lemma 10.
   b. $\vdash |1\rangle ?{\cdot} : S(\Psi_2 \Rightarrow C)$, and $\vdash u : S(\Psi_2)$, with $S(C) \preceq \Psi_1 \Rightarrow B$, but this case is impossible by Lemma 10.
2. $\vdash |1\rangle ?u{\cdot} : S(\Psi_1 \Rightarrow B)$ and $\vdash v : S(\Psi_1)$, with $S(B) \preceq A$. Then, by Lemma 5 again, one of the following possibilities happens:
   a. $\vdash |1\rangle ?{\cdot} : \Psi_2 \Rightarrow C$, and $\vdash u : \Psi_2$, with $CE \preceq S(\Psi_1 \Rightarrow B) \preceq S(\Psi_1 \Rightarrow A)$. Then, by Lemma 5 again, one of the following possibilities happens:
      i. $\vdash ?{\cdot} : \Psi_3 \Rightarrow D$, and $\vdash |1\rangle : \Psi_3$, with $D \preceq \Psi_2 \Rightarrow C \preceq \Psi_2 \Rightarrow S(\Psi_1 \Rightarrow A)$. Then, by Lemma 5 again, $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq \Psi_3 \Rightarrow \Psi_2 \Rightarrow S(\Psi_1 \Rightarrow A)$, so $\Psi_2 = \mathbb{Q} \preceq A$, and hence, by rule $\preceq$, $\vdash u : A$.
      ii. $\vdash ?{\cdot} : S(\Psi_3 \Rightarrow D)$, and $\vdash |1\rangle : S(\Psi_3)$, with $S(D) \preceq \Psi_2 \Rightarrow C$, but this case is impossible by Lemma 10.
   b. $\vdash |1\rangle ?{\cdot} : S(\Psi_2 \Rightarrow C)$, and $\vdash u : S(\Psi_2)$, with $S(C) \preceq S(\Psi_1 \Rightarrow B)$ Then, by Lemma 5 again, one of the following possibilities happens:
      i. $\vdash ?{\cdot} : \Psi_3 \Rightarrow D$, and $\vdash |1\rangle : \Psi_3$, with $D \preceq S(\Psi_2 \Rightarrow C) \preceq S(\Psi_2 \Rightarrow S(\Psi_1 \Rightarrow A))$. Then, by lemma 5 again, $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq \Psi_3 \Rightarrow D \preceq \Psi_3 \Rightarrow S(\Psi_2 \Rightarrow S(\Psi_1 \Rightarrow A))$. So, $\Psi_2 = \mathbb{Q} \preceq A$, hence, by rule $\preceq$, $\vdash u : A$.
      ii. $\vdash ?{\cdot} : S(\Psi_3 \Rightarrow D)$, and $\vdash |1\rangle : S(\Psi_3)$, with $S(D) \preceq S(\Psi_2 \Rightarrow C)$. By Lemma 5 again, $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq S(\Psi_3 \Rightarrow D) \preceq S(\Psi_3 \Rightarrow S(\Psi_2 \Rightarrow S(\Psi_2 \Rightarrow A)))$, so $\Psi_2 = \mathbb{Q} \preceq A$, and hence, by rule $\preceq$, $\vdash u : A$.

(if$_0$) Analogous to case (if$_1$).

(lin$_r^+$) Let $\vdash t(u + v) : A$, with $\vdash t : Q \Rightarrow B$. Then, by Lemma 5, one of the following cases happens:

1. $\vdash t : \Psi \Rightarrow C$ and $\vdash (u+v) : \Psi$, with $C \preceq A$. However, since $\vdash t : Q \Rightarrow B$, we have $\Psi \in \mathcal{B}$, which is impossible due to Corollary 6.
2. $\vdash t : S(\Psi \Rightarrow C)$ and $\vdash (u+v) : S(\Psi)$, with $S(C) \preceq A$. Then, by Corollary 6, $\vdash u : S(\Psi)$ and $\vdash v : S(\Psi)$. Hence,

$$\dfrac{\dfrac{\dfrac{\vdash t : S(\Psi \Rightarrow C) \quad \vdash u : S(\Psi)}{\vdash tu : S(C)} \Rightarrow_{ES} \quad \dfrac{\vdash t : S(\Psi \Rightarrow C) \quad \vdash v : S(\Psi)}{\vdash tv : S(C)} \Rightarrow_{ES}}{\vdash (tu+tv) : S(S(C))} S_I^+}{\vdash (tu+tv) : A} \preceq$$

($\mathsf{lin}_\mathsf{r}^\alpha$) Let $\vdash t(\alpha.u) : A$, with $\vdash t : Q \Rightarrow B$. Then, by Lemma 5, one of the following cases happens:

1. $\vdash t : \Psi \Rightarrow C$ and $\vdash \alpha.u : \Psi$, with $C \preceq A$. However, since $\vdash t : Q \Rightarrow B$, we have $\Psi \in \mathcal{B}$, which is impossible due to Corollary 6.
2. $\vdash t : S(\Psi \Rightarrow C)$ and $\vdash \alpha.u : S(\Psi)$, with $S(C) \preceq A$. Then, by Corollary 6, $\vdash u : S(\Psi)$. Hence,

$$\dfrac{\dfrac{\dfrac{\vdash t : S(\Psi \Rightarrow C) \quad \vdash u : S(\Psi)}{\vdash tu : S(C)} \Rightarrow_{ES}}{\vdash \alpha.tu : S(S(C))} S_I^\alpha}{\vdash \alpha.tu : A} \preceq$$

($\mathsf{lin}_\mathsf{r}^0$) Let $\vdash t\vec{0}_{S(Q)} : B$, with $\vdash t : Q \Rightarrow A$. Then, by Lemma 5, one of the following cases happens:

1. $\vdash t : \Psi \Rightarrow C$ and $\vdash \vec{0}_{S(A)} : \Psi$, with $C \preceq A$. Then, by Lemma 5 again, $S(A) \preceq \Psi$. However, since $\vdash t : Q \Rightarrow B$, $\Psi \in \mathcal{B}$, which is impossible by Lemma 10.
2. $\vdash t : S(\Psi \Rightarrow C)$ and $\vdash \vec{0}_{S(A)} : S(\Psi)$, with $S(C) \preceq A$. By rule $Ax_{\vec{0}}$, $\vdash \vec{0}_{S(A)} : S(C)$, hence we conclude by rule $\preceq$.

($\mathsf{lin}_\mathsf{l}^+$) Let $\vdash (t+u)v : A$. Then by Lemma 5, one of the following cases happens:

1. $\vdash (t+u) : \Psi \Rightarrow B$, which is impossible by Corollary 6.
2. $\vdash (t+u) : S(\Psi \Rightarrow B)$ and $\vdash v : S(\Psi)$, with $S(B) \preceq A$. Then, by Corollary 6, $\vdash t : S(\Psi \Rightarrow B)$ and $\vdash u : S(\Psi \Rightarrow B)$. Hence,

$$\dfrac{\dfrac{\dfrac{\vdash t : S(\Psi \Rightarrow B) \quad \vdash v : S(\Psi)}{\vdash tv : S(B)} \Rightarrow_{ES} \quad \dfrac{\vdash u : S(\Psi \Rightarrow B) \quad \vdash v : S(\Psi)}{\vdash uv : S(B)} \Rightarrow_{ES}}{\vdash (tv+uv) : S(S(B))} S_I^+}{\vdash (tv+uv) : A} \preceq$$

($\mathsf{lin}_\mathsf{l}^\alpha$) Let $\vdash (\alpha.t)u : A$. Then, by Lemma 5, one of the following cases happens:

1. $\vdash \alpha.t : \Psi \Rightarrow B$, which is impossible by Corollary 6.
2. $\vdash \alpha.t : S(\Psi \Rightarrow B)$ and $\vdash u : S(\Psi)$, with $S(B) \preceq A$. Then, by Corollary 6, $\vdash t : S(\Psi \Rightarrow B)$. Hence,

$$\dfrac{\dfrac{\dfrac{\vdash t : S(\Psi \Rightarrow B) \quad \vdash u : S(\Psi)}{\vdash tu : S(B)} \Rightarrow_{ES}}{\vdash \alpha.tu : S(S(B))} S_I^\alpha}{\vdash \alpha.tu : A} \preceq$$

($\mathsf{lin}_\mathsf{l}^0$) Let $\vdash \vec{0}_{S(Q \Rightarrow B)}t : A$. Then, by Lemma 5, one of the following cases happens:

1. $\vdash \vec{0}_{S(Q \Rightarrow B)} : \Psi \Rightarrow C$ and $\vdash t : \Psi$, with $C \preceq A$. Then, by Lemma 5 again, $S(Q \Rightarrow B) \preceq \Psi \Rightarrow C$, which is impossible by Lemma 10.

2. $\vdash \vec{0}_{S(Q\Rightarrow B)} : S(\Psi \Rightarrow C)$ and $\vdash t : S(\Psi)$, with $S(C) \preceq A$. By Lemma 5 again, $S(Q \Rightarrow B) \preceq S(\Psi \Rightarrow C)$. Hence, $B \preceq C$, and then $S(B) \preceq S(C) \preceq A$. By rule $Ax_{\vec{0}}$, $\vdash \vec{0}_{S(B)} : S(B)$, hence we conclude by rule $\preceq$.

(dist$_r^+$) Let $\vdash \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} ((r+s)\otimes u) : A$. By Lemma 5, $S(B\otimes C) \preceq A$ and $\vdash (r+s)\otimes u : S(S(B)\otimes C)$. Then, by the same Lemma, $\vdash (r+s) : D$ and $\vdash u : E$, with $D\otimes E \preceq S(S(B)\otimes C)$, so $D \preceq S(B)$ and $E \preceq C$, and hence, $\vdash (r+s) : S(B)$ and $\vdash u : C$. Then, by Corollary 6, $\vdash r : S(B)$ and $\vdash s : S(B)$. Therefore,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash r : S(B) \quad \vdash u : C}{\vdash r\otimes u : S(B)\otimes C}\otimes_I}{\vdash r\otimes u : S(S(B)\otimes C)}\preceq}{\vdash \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (r\otimes u) : S(B\otimes C)}\Uparrow_r \quad \dfrac{\dfrac{\dfrac{\vdash s : S(B) \quad \vdash u : C}{\vdash s\otimes u : S(B)\otimes C}\otimes_I}{\vdash s\otimes u : S(S(B)\otimes C)}\preceq}{\vdash \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (s\otimes u) : S(B\otimes C)}\Uparrow_r}{\vdash (\Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (r\otimes u) + \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (s\otimes u)) : S(S(B\otimes C))}S_I^+}{\vdash (\Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (r\otimes u) + \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (s\otimes u)) : A}\preceq$$

(dist$_l^+$) Analogous to case (dist$_r^+$).

(dist$_r^\alpha$) Let $\vdash \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} ((\alpha.r)\otimes u) : A$. By Lemma 5, $S(B\otimes C) \preceq A$, and $\vdash ((\alpha.r)\otimes u) : S(S(B)\otimes C)$. Then, by the same Lemma, $\vdash \alpha.r : D$ and $\vdash u : E$, with $D\otimes E \preceq S(S(B)\otimes C)$. Hence, $D \preceq S(B)$ and $E \preceq C$, so by rule $\preceq$, $\vdash \alpha.r : S(B)$ and $\vdash u : C$. By Corollary 6, $\vdash r : S(B)$. Therefore,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash r : S(B) \quad \vdash u : C}{\vdash r\otimes u : S(B)\otimes C}\otimes_I}{\vdash r\otimes u : S(S(B)\otimes C)}\preceq}{\vdash \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (r\otimes u) : S(B\otimes C)}\Uparrow_r}{\vdash \alpha.\Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (r\otimes u) : S(S(B\otimes C))}S_I^\alpha}{\vdash \alpha.\Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (r\otimes u) : A}\preceq$$

(dist$_l^\alpha$) Analogous to case (dist$_r^\alpha$).

(dist$_r^0$) Let $\vdash \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (\vec{0}_{S(B)}\otimes u) : A$. By Lemma 5, $S(B\otimes C) \preceq A$. By rule $Ax_{\vec{0}}$, $\vdash \vec{0}_{S(B\otimes C)} : S(B\otimes C)$. Hence, we conclude by rule $\preceq$.

(dist$_l^0$) Analogous to case (dist$_r^0$).

(dist$_\Uparrow^+$) Let $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} (t+u) : A$. Then, by Lemma 5, $S(C\otimes D) \preceq A$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} t : S(C\otimes D)$ and $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} u : S(C\otimes D)$. We conclude by rules S_I^+ and $\preceq$.

(dist$_\Uparrow^\alpha$) Let $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} (\alpha.t) : A$. Then, by Lemma 5, $S(C\otimes D) \preceq A$, and $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} t : S(C\otimes D)$. We conclude by rules S_I^α and $\preceq$.

(neut$_r^\Uparrow$) Let $\vdash \Uparrow_{S(S(B)\otimes C)}^{S(B\otimes C)} (b\otimes r) : A$, with $b \in \mathcal{B}$. Then, by Lemma 5, $\vdash b\otimes r : S(S(B)\otimes C)$ and $S(B\otimes C) \preceq A$. Then, by Lemma 5 again, $\vdash b : D$ and $\vdash r : E$, with $D\otimes E \preceq S(S(B)\otimes C)$, so $D \preceq S(B)$ and $E \preceq C$, hence, $\vdash b : S(B)$ and $\vdash r : E$. Therefore, by Corollary 11, $\vdash b : B$, and so, by rule $\otimes_I$, $\vdash b\otimes r : B\otimes C$, and by rule $\preceq$, $\vdash b\otimes r : S(B\otimes C)$.

(neut$_l^\Uparrow$) Analogous to case (neut$_r^\Uparrow$).

(proj) Let $\vdash \pi_j(\sum_{i=1}^n [\alpha_i.](b_{1i}\otimes\cdots\otimes b_{mi})) : A$. Then, by Lemma 5, we have that $\vdash \sum_{i=1}^n [\alpha_i.](b_{1i}\otimes\cdots\otimes b_{mi}) : Q^S$, with $Q^{S\setminus\{1,\dots,j\}} \preceq A$. Then, by Corollary 6, we have for every i, $\vdash b_{1i}\otimes\cdots\otimes b_{mi} : Q^S$. By Lemma 5 again, for every k, $\vdash b_{ki} : B_k$, such that $\bigotimes_k B_k \preceq Q^S$. Since b_{ki} are either $|0\rangle$ or $|1\rangle$, by Lemma 5, for every k, $\mathbb{Q} \preceq B_k$. Then, by rules $\preceq$, $\otimes_I$, α, S_I^+ and $\otimes_I$ again, we can derive $\vdash \bigotimes_{t=1}^j b_{tk} \otimes \sum_{i\in P}\left(\frac{\alpha_i}{\sqrt{\sum_{i\in P}|\alpha_i|^2}}\right).(b_{j+1,i}\otimes\cdots\otimes b_{mi}) : Q^{S\setminus\{1,\dots,j\}}$, for any k and P such that $\forall i\in P$, $\forall t\le j$, $b_{ti} = b_{tk}$.

(comm) Let $\vdash (u+v) : A$. By Lemma 5, $\vdash u : B$ and $\vdash v : B$, with $S(B) \preceq A$. So,

$$\dfrac{\dfrac{\vdash v : B \qquad \vdash u : B}{\vdash (v+u) : S(B)}\ S_I^+}{\vdash (v+u) : A}\ \preceq$$

(assoc) Let $\vdash ((u+v)+w) : A$. By Lemma 5, $\vdash (u+v) : B$ and $\vdash w : B$, with $S(B) \preceq A$. Then, by Corollary 6, $\vdash u : B$ and $\vdash v : B$. Hence,

$$\dfrac{\dfrac{\dfrac{\vdash u : B}{\vdash u : S(B)}\ \preceq \qquad \dfrac{\vdash v : B \quad \vdash w : B}{\vdash (v+w) : S(B)}\ S_I^+}{\vdash (u+(v+w)) : S(S(B))}\ S_I^+}{\vdash (u+(v+w)) : A}\ \preceq$$

Contextual rules Let $t \longrightarrow_{(p)} u$. Then,

($tv \longrightarrow_{(p)} uv$) Let $\vdash tv : A$. By Lemma 5, one of the following cases happens:
- $\vdash t : \Psi \Rightarrow B$ and $\vdash v : \Psi$, with $B \preceq A$. Then, by the induction hypothesis, $\vdash u : \Psi \Rightarrow B$. We conclude by rules $\Rightarrow_E$ and $\preceq$.
- $\vdash t : S(\Psi \Rightarrow B)$ and $\vdash v : S(\Psi)$, with $S(B) \preceq A$. Then, by the induction hypothesis, $\vdash u : S(\Psi \Rightarrow B)$. We conclude by rules $\Rightarrow_{ES}$ and $\preceq$.

($(\lambda x : Q\ v)t \longrightarrow_{(p)} (\lambda x : Q\ v)u$) Let $\vdash (\lambda x : Q\ v)t : A$. By Lemma 5, one of the following cases happens:
- $\vdash (\lambda x : Q\ v) : \Psi \Rightarrow B$ and $\vdash t : \Psi$, with $B \preceq A$. Then, by the induction hypothesis, $\vdash u : \Psi$. We conclude by rules $\Rightarrow_E$ and $\preceq$.
- $\vdash (\lambda x : Q\ v) : S(\Psi \Rightarrow B)$ and $\vdash t : S(\Psi)$, with $S(B) \preceq A$. Then, by the induction hypothesis, $\vdash u : S(\Psi)$. We conclude by rules $\Rightarrow_{ES}$ and $\preceq$.

($\pi_j t \longrightarrow_{(p)} \pi_j u$) Let $\vdash \pi_j t : A$. By Lemma 5, $\vdash t : Q_n^S$, where $Q_n^{\{1,\dots,j\}} \preceq A$. Then, by the induction hypothesis, $\vdash u : Q_n^S$. We conclude by rules S_E and $\preceq$.

($(t+v) \longrightarrow_{(p)} (u+v)$) Let $\vdash (t+v) : A$. By Lemma 5, $\vdash t : B$ and $\vdash v : B$, with $S(B) \preceq A$. Then, by the induction hypothesis, $\vdash u : B$. We conclude by rules S_I^+ and $\preceq$.

($\alpha.t \longrightarrow_{(p)} \alpha.u$) Let $\vdash \alpha.t : A$. By Lemma 5, $\vdash t : S(B)$, with $S(B) \preceq A$. Then, by the induction hypothesis, $\vdash u : S(B)$. We conclude by rules S_I^α and $\preceq$.

($t \otimes v \longrightarrow_{(p)} u \otimes v$) Let $\vdash t \otimes v : A$. By Lemma 5, $\vdash t : B$ and $\vdash v : C$, with $B \otimes C \preceq A$. Then, by the induction hypothesis, $\vdash u : B$. We conclude by rules $\otimes_I$ and $\preceq$.

($v \otimes t \longrightarrow_{(p)} v \otimes u$) Analogous to previous case.

(*head* $t \longrightarrow_{(p)}$ *head* u) Let $\vdash$ *head* $t : A$. By Lemma 5, $\vdash t : Q \otimes Q$, with $Q \preceq A$. Then, by the induction hypothesis, $\vdash u : Q \otimes Q$. We conclude by rules $\otimes_{Er}$ and $\preceq$.

(*tail* $t \longrightarrow_{(p)}$ *tail* u) Let $\vdash$ *tail* $t : A$. By Lemma 5, $\vdash t : Q \otimes Q$, with $Q \preceq A$. Then, by the induction hypothesis, $\vdash u : Q \otimes Q$. We conclude by rules $\otimes_{Er}$ and $\preceq$.

($\Uparrow_{S(B)}^{S(C\otimes D)} t \longrightarrow_{(p)} \Uparrow_{S(B)}^{S(C\otimes D)} u$) Let $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} t : A$. By Lemma 5, $\vdash t : S(B)$, $S(C \otimes D) \preceq A$, where either:
- a rule R allows typing $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} t : S(C \otimes D)$ starting from $\vdash t : S(B)$. Then, by the induction hypothesis, $\vdash u : S(B)$, and using rule R, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} u : S(C \otimes D)$. We conclude by rule $\preceq$.
- $t = \alpha.t'$ and $\vdash t' : S(B)$. The possible reductions $t \longrightarrow_{(p)} u$ are:
  - $u = \alpha.u'$. By the induction hypothesis, $\vdash u' : S(B)$, and so by some rule $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} u' : S(C \otimes D)$. We conclude by rule $\Uparrow^\alpha$.
  - $\alpha = 1$ and $u = t'$, then by some rule $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} t' : S(C \otimes D)$.

- $\alpha = 0$ and $u = \vec{0}_{S(B)}$. By rule $Ax_{\vec{0}}$ $\vdash \vec{0}_{S(B)} : S(B)$, and by one of the rules $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} \vec{0}_{S(B)} : S(C \otimes D)$.
- $t' = u = \vec{0}_{S(B)}$, then, by one of the rules $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} \vec{0}_{S(B)} : S(C \otimes D)$.
- $t' = \beta.t''$ and $u = (\alpha \times \beta).t''$. By Corollary 6, $\vdash t'' : S(B)$. Then, by one of the rules $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} t'' : S(C \otimes D)$, and by rule $\Uparrow^\alpha$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} (\alpha \times \beta).t'' : S(C \otimes D)$.
- $t' = (t_1 + t_2)$ and $u = (\alpha.t_1 + \alpha.t_2)$. By the induction hypothesis $\vdash (\alpha.t_1 + \alpha.t_2) : S(B)$, then, by one of the rules $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} (\alpha.t_1 + \alpha.t_1) : S(C \otimes D)$.

In any case, we conclude by rule $\preceq$.

- $t = (t_1 + t_2)$ and for $i = 1, 2$, $\vdash t_i : S(B)$. The possible reductions $(t_1 + t_2) \longrightarrow_{(p)} u$ are:
  - $u = (u_1 + u_2)$, with $t_1 = u_1$ and $t_2 \longrightarrow_{(p)} u_2$ or $t_1 \longrightarrow_{(p)} u_1$ and $t_2 = u_2$. Then, by the induction hypothesis $\vdash u_i : S(B)$, so by a rule $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} u_i : S(C \otimes D)$ and by rule $\Uparrow^+$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} (u_1 + u_2) : S(C \otimes D)$.
  - $t_1 = \vec{0}_{S(B)}$ and $u = t_2$. By some rule $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} t_2 : S(C \otimes D)$.
  - $t_1 = \alpha.t'$, $t_2 = \beta.t'$ and $u = (\alpha+\beta).t'$. By Corollary 6, $\vdash t' : S(B)$. Then, by some rule $\Uparrow$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} t' : S(C \otimes D)$, and by rule $\Uparrow^\alpha$, $\vdash \Uparrow_{S(B)}^{S(C\otimes D)} (\alpha + \beta).t' : S(C \otimes D)$.
  - $t_1 = \alpha.t_2$ and $u = (\alpha + 1).t_2$. Analogous to previous case.
  - $t_1 = t_2$ and $u = 2.t_1$. Analogous to previous case.

  In any case, we conclude by rule $\preceq$. ◀

C Trace and typing of the Deutsch algorithm

As usual in quantum computing, we may use $|q_1 \cdots q_n\rangle$ as a shorthand notation for $|q_1\rangle \otimes \cdots \otimes |q_n\rangle$.

The full trace of $\mathsf{Deutsch}_{id}$ is given below. $\Uparrow_2$ stands for $\Uparrow_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}^{S(\mathbb{Q}\otimes\mathbb{Q})}$ and $\Uparrow_{12}^2$ for $\Uparrow_{S(S(\mathbb{Q})\otimes S(\mathbb{Q}))}^{S(\mathbb{Q}\otimes S(\mathbb{Q}))}$.

$$
\begin{aligned}
&\mathsf{Deutsch}_{id}\\
&= \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_f\ \ \Uparrow_2\Uparrow_{12}^2\ (\mathsf{H}_{both}\ \ |01\rangle)))\\
&\xrightarrow{(\beta_b)}_{(1)} \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_2\Uparrow_{12}^2\ ((\mathsf{H}(head\ \ |01\rangle)) \otimes (\mathsf{H}(tail\ \ |01\rangle)))))\\
&\xrightarrow{(\mathsf{head})}_{(1)} \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_2\Uparrow_{12}^2\ ((\mathsf{H}\,|0\rangle) \otimes (\mathsf{H}(tail\ \ |01\rangle)))))\\
&\xrightarrow{(\mathsf{tail})}_{(1)} \pi_1\ \ \Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_2\Uparrow_{12}^2\ ((\mathsf{H}\,|0\rangle) \otimes (\mathsf{H}\,|1\rangle)))\\
&\xrightarrow{(\beta_b)^2}_{(1)} \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_2\Uparrow_{12}^2\ (\frac{1}{\sqrt{2}}.(|0\rangle + (|0\rangle?(-|1\rangle)\cdot|1\rangle)) \otimes \frac{1}{\sqrt{2}}.(|0\rangle + (|1\rangle?(-|1\rangle)\cdot|1\rangle)))))\\
&\xrightarrow{(\mathsf{if}_0)}_{(1)} \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_2\Uparrow_{12}^2\ (\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}.(|0\rangle + (|1\rangle?(-|1\rangle)\cdot|1\rangle)))))\\
&\xrightarrow{(\mathsf{if}_1)}_{(1)} \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_2\Uparrow_{12}^2\ (\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle))))\\
&\xrightarrow{(\mathsf{dist}_r^\alpha)}_{(1)} \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \frac{1}{\sqrt{2}}.\ \Uparrow_{S(S(\mathbb{Q})\otimes S(\mathbb{Q}))}^{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ ((|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle))))\\
&\xrightarrow{(\mathsf{dist}_r^+)}_{(1)} \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \frac{1}{\sqrt{2}}.\ \ (\Uparrow_{S(S(\mathbb{Q})\otimes S(\mathbb{Q}))}^{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |0\rangle \otimes \frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle)\\
&\qquad\qquad + \Uparrow_{S(S(\mathbb{Q})\otimes S(\mathbb{Q}))}^{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |1\rangle \otimes \frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle)))))\\
&\xrightarrow{(\mathsf{neut}_r^\Uparrow)}_{(1)} \pi_1\ (\Uparrow_{S(S(\mathbb{Q})\otimes\mathbb{Q})}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ \Uparrow_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}^{S(\mathbb{Q}\otimes\mathbb{Q})}\ \frac{1}{\sqrt{2}}.\ \ (|0\rangle \otimes \frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle)\\
&\qquad\qquad + |1\rangle \otimes \frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle)))))
\end{aligned}
$$

$$
\begin{aligned}
&\xrightarrow{(\mathsf{dist}^{\alpha}_{\Uparrow})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \frac{1}{\sqrt{2}}.\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|0\rangle\otimes\frac{1}{\sqrt{2}}.(|0\rangle-|1\rangle)\\
&\qquad\qquad +|1\rangle\otimes\frac{1}{\sqrt{2}}.(|0\rangle-|1\rangle)))))\\
&\xrightarrow{(\mathsf{dist}^{+}_{\Uparrow})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \frac{1}{\sqrt{2}}.\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|0\rangle\otimes\frac{1}{\sqrt{2}}.(|0\rangle-|1\rangle))\\
&\qquad\qquad +\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|1\rangle\otimes\frac{1}{\sqrt{2}}.(|0\rangle-|1\rangle))))))\\
&\xrightarrow{(\mathsf{dist}^{\alpha}_{\Uparrow})^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \frac{1}{\sqrt{2}}.\ (\frac{1}{\sqrt{2}}.\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|0\rangle\otimes(|0\rangle-|1\rangle))\\
&\qquad\qquad +\frac{1}{\sqrt{2}}.\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|1\rangle\otimes(|0\rangle-|1\rangle))))))\\
&\xrightarrow{(\alpha\mathsf{dist})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ (\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}.\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|0\rangle\otimes(|0\rangle-|1\rangle))\\
&\qquad\qquad +\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}.\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|1\rangle\otimes(|0\rangle-|1\rangle))))))\\
&\xrightarrow{(\mathsf{prod})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ (\frac{1}{2}.\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|0\rangle\otimes(|0\rangle-|1\rangle))\\
&\qquad\qquad +\frac{1}{2}.\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ (|1\rangle\otimes(|0\rangle-|1\rangle))))))\\
&\xrightarrow{(\mathsf{dist}^{+}_{\Uparrow})^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ (\frac{1}{2}.(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |00\rangle+\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |0\rangle\otimes(-|1\rangle))\\
&\qquad\qquad +\frac{1}{2}.(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |10\rangle+\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |1\rangle\otimes(-|1\rangle))))))\\
&\xrightarrow{(\mathsf{dist}^{\alpha}_{\Uparrow})^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \ (\frac{1}{2}.(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |00\rangle-\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |01\rangle)\\
&\qquad\qquad +\frac{1}{2}.(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |10\rangle-\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}))}\ |11\rangle)))))\\
&\xrightarrow{(\mathsf{neut}^{\Uparrow}_{\Downarrow})^4}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ (\frac{1}{2}.(|00\rangle-|01\rangle)+\frac{1}{2}.(|10\rangle-|11\rangle)))))\\
&\xrightarrow{(\mathsf{lin}^{+}_{\mathsf{r}})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\mathsf{U}_{id}\ \frac{1}{2}.(|00\rangle-|01\rangle)+\mathsf{U}_{id}\ \frac{1}{2}.(|10\rangle-|11\rangle))))\\
&\xrightarrow{(\mathsf{lin}^{\alpha}_{\mathsf{r}})^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.\mathsf{U}_{id}(|00\rangle-|01\rangle)+\frac{1}{2}.\mathsf{U}_{id}(|10\rangle-|11\rangle))))\\
&\xrightarrow{(\mathsf{lin}^{+}_{\mathsf{r}})^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.(\mathsf{U}_{id}\,|00\rangle+\mathsf{U}_{id}(-|01\rangle))+\frac{1}{2}.(\mathsf{U}_{id}\,|10\rangle+\mathsf{U}_{id}(-|11\rangle)))))\\
&\xrightarrow{(\mathsf{lin}^{\alpha}_{\mathsf{r}})^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.(\mathsf{U}_{id}\,|00\rangle-\mathsf{U}_{id}\,|01\rangle)+\frac{1}{2}.(\mathsf{U}_{id}\,|10\rangle-\mathsf{U}_{id}\,|11\rangle))))\\
&\xrightarrow{(\beta_{\mathsf{b}})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.((head\,|00\rangle)\otimes((tail\,|00\rangle)?(\mathsf{not}(id(head\,|00\rangle)))\cdot(id(head\,|00\rangle)))\\
&\qquad\qquad -\mathsf{U}_{id}\ \ |01\rangle)+\frac{1}{2}.(\mathsf{U}_{id}\ \ |10\rangle-\mathsf{U}_{id}\ \ |11\rangle))))\\
&\xrightarrow{(\mathsf{head})^3}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.(|0\rangle\otimes((tail\,|00\rangle)?(\mathsf{not}(id\,|0\rangle)))\cdot(id\,|0\rangle))\\
&\qquad\qquad -\mathsf{U}_{id}\ \ |01\rangle)+(\frac{1}{2}.(\mathsf{U}_{id}\ \ |10\rangle-\mathsf{U}_{id}\ \ |11\rangle))))\\
&\xrightarrow{(\mathsf{tail})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.(|0\rangle\otimes(|0\rangle?(\mathsf{not}(id\,|0\rangle)))\cdot(id\,|0\rangle))\\
&\qquad\qquad -\mathsf{U}_{id}\ \ |01\rangle)+(\frac{1}{2}.(\mathsf{U}_{id}\ \ |10\rangle-\mathsf{U}_{id}\ \ |11\rangle))))
\end{aligned}
$$

$$\xrightarrow{(\mathsf{if}_0)}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.(|0\rangle \otimes (id\,|0\rangle) - \mathsf{U}_{id}\ \ |01\rangle) + \frac{1}{2}.(\mathsf{U}_{id}\ \ |10\rangle - \mathsf{U}_{id}\ \ |11\rangle)))$$

$$\xrightarrow{(\beta_b)}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.(|00\rangle - \mathsf{U}_{id}\ \ |01\rangle) + \frac{1}{2}.(\mathsf{U}_{id}\ \ |10\rangle - \mathsf{U}_{id}\ \ |11\rangle)))$$

$$\longrightarrow^*_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (\frac{1}{2}.(|00\rangle - |01\rangle) + \frac{1}{2}.(|11\rangle - |10\rangle)))$$

$$\xrightarrow{(\mathsf{lin}_r^+)}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\mathsf{H}_1\ \frac{1}{2}.(|00\rangle - |01\rangle) + \mathsf{H}_1\ \frac{1}{2}.(|11\rangle - |10\rangle)))$$

$$\xrightarrow{(\mathsf{lin}_r^\alpha)^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{2}.\mathsf{H}_1(|00\rangle - |01\rangle) + \frac{1}{2}.\mathsf{H}_1(|11\rangle - |10\rangle)))$$

$$\xrightarrow{(\mathsf{lin}_r^+)^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{2}.(\mathsf{H}_1\,|00\rangle + \mathsf{H}_1(-\,|01\rangle)) + \frac{1}{2}.(\mathsf{H}_1\,|11\rangle + \mathsf{H}_1(-\,|10\rangle))))$$

$$\xrightarrow{(\mathsf{lin}_r^\alpha)^2}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{2}.(\mathsf{H}_1\,|00\rangle - \mathsf{H}_1\,|01\rangle) + \frac{1}{2}.(\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle)))$$

$$\xrightarrow{(\beta_b)}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{2}.((\mathsf{H}(head\,|00\rangle)) \otimes (tail\,|00\rangle) - \mathsf{H}_1\,|01\rangle) + \frac{1}{2}.(\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle)))$$

$$\xrightarrow{(\mathsf{head})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{2}.((\mathsf{H}\,|0\rangle) \otimes (tail\,|00\rangle) - \mathsf{H}_1\,|01\rangle) + \frac{1}{2}.(\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle)))$$

$$\xrightarrow{(\mathsf{tail})}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{2}.((\mathsf{H}\,|0\rangle) \otimes |0\rangle - \mathsf{H}_1\,|01\rangle) + \frac{1}{2}.(\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle)))$$

$$\xrightarrow{(\beta_b)}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{2}.((\frac{1}{\sqrt{2}}.(|0\rangle + |0\rangle?(-\,|1\rangle)\cdot|1\rangle)) \otimes |0\rangle - \mathsf{H}_1\,|01\rangle) + \frac{1}{2}.(\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle)))$$

$$\xrightarrow{(\mathsf{if}_0)}_{(1)} \pi_1\ (\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{2}.((\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle)) \otimes |0\rangle - \mathsf{H}_1\,|01\rangle) + \frac{1}{2}.(\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle)))$$

$$\xrightarrow{(\mathsf{dist}_\Uparrow^+)}_{(1)} \pi_1(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \frac{1}{2}.((\frac{1}{\sqrt{2}}.\,|0\rangle + \frac{1}{\sqrt{2}}.\,|1\rangle) \otimes |0\rangle - \mathsf{H}_1\,|01\rangle) + \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \frac{1}{2}.(\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle))$$

$$\xrightarrow{(\mathsf{dist}_\Uparrow^\alpha)^2}_{(1)} \pi_1(\frac{1}{2}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ ((\frac{1}{\sqrt{2}}.\,|0\rangle + \frac{1}{\sqrt{2}}.\,|1\rangle) \otimes |0\rangle - \mathsf{H}_1\,|01\rangle) + \frac{1}{2}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle))$$

$$\xrightarrow{(\mathsf{dist}_\Uparrow^+)}_{(1)} \pi_1(\frac{1}{2}.(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ ((\frac{1}{\sqrt{2}}.\,|0\rangle + \frac{1}{\sqrt{2}}.\,|1\rangle) \otimes |0\rangle) + \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (-\mathsf{H}_1\,|01\rangle))$$
$$+\frac{1}{2}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle))$$

$$\xrightarrow{(\mathsf{dist}_\Uparrow^\alpha)}_{(1)} \pi_1(\frac{1}{2}.(\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ ((\frac{1}{\sqrt{2}}.\,|0\rangle + \frac{1}{\sqrt{2}}.\,|1\rangle) \otimes |0\rangle) - \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\,|01\rangle)$$
$$+\frac{1}{2}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle))$$

$$\xrightarrow{(\mathsf{dist}_r^+)}_{(1)} \pi_1(\frac{1}{2}.((\Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{\sqrt{2}}.\,|0\rangle) \otimes |0\rangle + \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\frac{1}{\sqrt{2}}.\,|1\rangle) \otimes |0\rangle) - \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\,|01\rangle)$$
$$+\frac{1}{2}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle))$$

$$\xrightarrow{(\mathsf{dist}_r^\alpha)^2}_{(1)} \pi_1(\frac{1}{2}.((\frac{1}{\sqrt{2}}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ |00\rangle + \frac{1}{\sqrt{2}}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ |10\rangle) - \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\,|01\rangle)$$
$$+\frac{1}{2}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle))$$

$$\xrightarrow{(\mathsf{neut}_r^\Uparrow)^2}_{(1)} \pi_1(\frac{1}{2}.((\frac{1}{\sqrt{2}}.\,|00\rangle + \frac{1}{\sqrt{2}}.\,|10\rangle) - \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\,|01\rangle) + \frac{1}{2}.\ \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ (\mathsf{H}_1\,|11\rangle - \mathsf{H}_1\,|10\rangle))$$

$$\longrightarrow^*_{(1)} \pi_1(\frac{1}{2}.(((\frac{1}{\sqrt{2}}.\,|00\rangle + \frac{1}{\sqrt{2}}.\,|10\rangle) + (-\frac{1}{\sqrt{2}}.\,|01\rangle - \frac{1}{\sqrt{2}}.\,|11\rangle))$$
$$+((\frac{1}{\sqrt{2}}.\,|01\rangle - \frac{1}{\sqrt{2}}.\,|11\rangle) + (-\frac{1}{\sqrt{2}}.\,|00\rangle + \frac{1}{\sqrt{2}}.\,|10\rangle))))$$

$$
\begin{aligned}
&=_{AC}\ \pi_1(\frac{1}{2}.(((\frac{1}{\sqrt{2}}.\,|00\rangle-\frac{1}{\sqrt{2}}.\,|00\rangle)+(-\frac{1}{\sqrt{2}}.\,|01\rangle+\frac{1}{\sqrt{2}}.\,|01\rangle))\\
&\qquad\qquad +((-\frac{1}{\sqrt{2}}.\,|11\rangle-\frac{1}{\sqrt{2}}.\,|11\rangle)+(\frac{1}{\sqrt{2}}.\,|10\rangle+\frac{1}{\sqrt{2}}.\,|10\rangle))))\\
&\xrightarrow{(\mathsf{fact})^4}_{(1)}\ \pi_1(\frac{1}{2}.(((\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}).\,|00\rangle+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}).\,|01\rangle)\\
&\qquad\qquad +((-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}).\,|11\rangle+(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}).\,|10\rangle)))\\
&=\ \ \pi_1(\frac{1}{2}.((0.\,|00\rangle+0.\,|01\rangle)+(-\frac{2}{\sqrt{2}}.\,|11\rangle+\frac{2}{\sqrt{2}}.\,|10\rangle)))\\
&\xrightarrow{(\mathsf{zero}_\alpha)^2}_{(1)}\ \pi_1(\frac{1}{2}.((\vec{0}_{S(\mathbb{Q}\otimes\mathbb{Q})}+\vec{0}_{S(\mathbb{Q}\otimes\mathbb{Q})})+(-\frac{2}{\sqrt{2}}.\,|11\rangle+\frac{2}{\sqrt{2}}.\,|10\rangle)))\\
&\xrightarrow{(\mathsf{neutral})^2}_{(1)}\ \pi_1(\frac{1}{2}.(-\frac{2}{\sqrt{2}}.\,|11\rangle+\frac{2}{\sqrt{2}}.\,|10\rangle))\\
&\xrightarrow{(\alpha\mathsf{dist})}_{(1)}\ \pi_1(\frac{1}{2}.(-\frac{2}{\sqrt{2}}).\,|11\rangle+\frac{1}{2}.\frac{2}{\sqrt{2}}.\,|10\rangle)\\
&\xrightarrow{(\mathsf{prod})^2}_{(1)}\ \pi_1(-\frac{1}{\sqrt{2}}.\,|11\rangle+\frac{1}{\sqrt{2}}.\,|10\rangle)\\
&=_{AC}\ \pi_1(\frac{1}{\sqrt{2}}.\,|10\rangle-\frac{1}{\sqrt{2}}.\,|11\rangle)\\
&\xrightarrow{(\mathsf{proj})}_{(1)}\ |1\rangle\otimes(\frac{1}{\sqrt{2}}.\,|0\rangle-\frac{1}{\sqrt{2}}.\,|1\rangle)
\end{aligned}
$$

The typing of $\mathsf{Deutsch}_f$, for any $\vdash f:\mathbb{Q}\Rightarrow\mathbb{Q}$, is given below:

$$
\dfrac{
\dfrac{\dfrac{}{\vdash|0\rangle:\mathbb{Q}}\,Ax_{|0\rangle}}{\vdash|0\rangle:S(\mathbb{Q})}\,\preceq
\quad
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\vdash ?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\mathit{If}\quad\dfrac{}{x:\mathbb{Q}\vdash x:\mathbb{Q}}\,Ax}{x:\mathbb{Q}\vdash x?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_E}{x:\mathbb{Q}\vdash x?\cdot:S(\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q})}\,\preceq
\quad
\dfrac{\dfrac{}{\vdash|1\rangle:\mathbb{Q}}\,Ax_{|1\rangle}}{\vdash-|1\rangle:S(\mathbb{Q})}\,S_I^\alpha
}{x:\mathbb{Q}\vdash x?(-|1\rangle)\cdot:S(\mathbb{Q}\Rightarrow\mathbb{Q})}\,\Rightarrow_{ES}
\quad
\dfrac{\dfrac{}{\vdash|1\rangle:\mathbb{Q}}\,Ax_{|1\rangle}}{\vdash|1\rangle:S(\mathbb{Q})}\,\preceq
}{x:\mathbb{Q}\vdash x?(-|1\rangle)\cdot|1\rangle:S(\mathbb{Q})}\,\Rightarrow_{ES}
}{
\dfrac{
\dfrac{
\dfrac{x:\mathbb{Q}\vdash|0\rangle+x?(-|1\rangle)\cdot|1\rangle:S(S(\mathbb{Q}))}{x:\mathbb{Q}\vdash\frac{1}{\sqrt{2}}.(|0\rangle+x?(-|1\rangle)\cdot|1\rangle):S(S(S(\mathbb{Q})))}\,S_I^\alpha
}{x:\mathbb{Q}\vdash\frac{1}{\sqrt{2}}.(|0\rangle+x?(-|1\rangle)\cdot|1\rangle):S(\mathbb{Q})}\,\preceq
}{\vdash\mathsf{H}:\mathbb{Q}\Rightarrow S(\mathbb{Q})}\,\Rightarrow_I
}\,S_I^+
\tag{2}
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{(2)}{\vdash\mathsf{H}:\mathbb{Q}\Rightarrow S(\mathbb{Q})}
\quad
\dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ x:\mathbb{Q}}\,E_r
}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{H}\ (\mathit{head}\ x):S(\mathbb{Q})}\,\Rightarrow_E
\quad
\dfrac{\dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash\mathit{tail}\ y:\mathbb{Q}}\,E_l
}{
\dfrac{
\dfrac{x:\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathsf{H}\ (\mathit{head}\ x))\otimes(\mathit{tail}\ y):S(\mathbb{Q})\otimes\mathbb{Q}}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathsf{H}\ (\mathit{head}\ x))\otimes(\mathit{tail}\ x):S(\mathbb{Q})\otimes\mathbb{Q}}\,C
}{\vdash\mathsf{H}_1:\mathbb{Q}\otimes\mathbb{Q}\Rightarrow S(\mathbb{Q})\otimes\mathbb{Q}}\,\Rightarrow_I
}\,\otimes_I
}{}
\tag{3}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash ?\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\mathit{If}\qquad \dfrac{}{x:\mathbb{Q}\vdash x:\mathbb{Q}}\,Ax}{x:\mathbb{Q}\vdash x?\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_E \qquad \dfrac{}{\vdash |0\rangle : \mathbb{Q}}\,Ax_{|0\rangle}}{x:\mathbb{Q}\vdash x?|0\rangle\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_E \qquad \dfrac{}{\vdash |1\rangle : \mathbb{Q}}\,Ax_{|1\rangle}}{x:\mathbb{Q}\vdash x?|0\rangle\cdot|1\rangle : \mathbb{Q}}\,\Rightarrow_E}{\vdash \mathsf{not} : \mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_I \tag{4}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash ?\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\mathit{If}_A \quad \dfrac{\dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{tail}\ y:\mathbb{Q}}\,\otimes_{E_l}}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathit{tail}\ y)?\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_E \quad \dfrac{\dfrac{(4)}{\vdash \mathsf{not}:\mathbb{Q}\Rightarrow\mathbb{Q}} \quad \dfrac{\dfrac{}{\vdash f:\mathbb{Q}\Rightarrow\mathbb{Q}} \quad \dfrac{\dfrac{}{z:\mathbb{Q}\otimes\mathbb{Q}\vdash z:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{z:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ z:\mathbb{Q}}\,\otimes_{E_r}}{z:\mathbb{Q}\otimes\mathbb{Q}\vdash f\ (\mathit{head}\ z):\mathbb{Q}}\,\Rightarrow_E}{z:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathsf{not}\ (f\ (\mathit{head}\ z)):\mathbb{Q}}\,\Rightarrow_E}{y:\mathbb{Q}\otimes\mathbb{Q},z:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathit{tail}\ y)?(\mathsf{not}\ (f\ (\mathit{head}\ z)))\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_E}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathit{tail}\ y)?(\mathsf{not}\ (f\ (\mathit{head}\ y)))\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}}\,C \quad \dfrac{\dfrac{}{\vdash f:\mathbb{Q}\Rightarrow\mathbb{Q}} \quad \dfrac{\dfrac{}{z:\mathbb{Q}\otimes\mathbb{Q}\vdash z:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{z:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ z:\mathbb{Q}}\,\otimes_{E_r}}{z:\mathbb{Q}\otimes\mathbb{Q}\vdash f\ (\mathit{head}\ z):\mathbb{Q}}\,\Rightarrow_E}{\dfrac{y:\mathbb{Q}\otimes\mathbb{Q},z:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathit{tail}\ y)?(\mathsf{not}\ (f\ (\mathit{head}\ y)))\cdot(f\ (\mathit{head}\ z)):\mathbb{Q}}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathit{tail}\ y)?(\mathsf{not}\ (f\ (\mathit{head}\ y)))\cdot(f\ (\mathit{head}\ y)):\mathbb{Q}}\,C}\,\Rightarrow_E \tag{5}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ x:\mathbb{Q}}\,\otimes_{E_r} \quad \dfrac{(5)}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathit{tail}\ y)?(\mathsf{not}\ (f\ (\mathit{head}\ y)))\cdot(f\ (\mathit{head}\ y)):\mathbb{Q}}}{x:\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathit{head}\ x)\otimes(\mathit{tail}\ y)?(\mathsf{not}\ (f\ (\mathit{head}\ y)))\cdot f\ (\mathit{head}\ y):\mathbb{Q}\otimes\mathbb{Q}}\,\otimes_I}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathit{head}\ x)\otimes(\mathit{tail}\ x)?(\mathsf{not}\ (f\ (\mathit{head}\ x)))\cdot f\ (\mathit{head}\ x):\mathbb{Q}\otimes\mathbb{Q}}\,C}{\vdash \mathsf{U}_f:\mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q}}\,\Rightarrow_I \tag{6}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(2)}{\vdash \mathsf{H}:\mathbb{Q}\Rightarrow S(\mathbb{Q})} \quad \dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ x:\mathbb{Q}}\,\otimes_{E_r}}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathsf{H}\ (\mathit{head}\ x):S(\mathbb{Q})}\,\Rightarrow_E \quad \dfrac{\dfrac{(2)}{\vdash \mathsf{H}:\mathbb{Q}\Rightarrow S(\mathbb{Q})} \quad \dfrac{\dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{tail}\ y:\mathbb{Q}}\,\otimes_{E_l}}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathsf{H}\ (\mathit{tail}\ y):S(\mathbb{Q})}\,\Rightarrow_E}{x:\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathsf{H}\ (\mathit{head}\ x))\otimes(\mathsf{H}\ (\mathit{tail}\ y)):S(\mathbb{Q})\otimes S(\mathbb{Q})}\,\otimes_I}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash (\mathsf{H}\ (\mathit{head}\ x))\otimes(\mathsf{H}\ (\mathit{tail}\ x)):S(\mathbb{Q})\otimes S(\mathbb{Q})}\,C}{\vdash \mathsf{H}_{both}:\mathbb{Q}\otimes\mathbb{Q}\Rightarrow S(\mathbb{Q})\otimes S(\mathbb{Q})}\,\Rightarrow_I \quad \dfrac{\dfrac{}{\vdash |0\rangle:\mathbb{Q}}\,Ax_{|0\rangle} \quad \dfrac{}{\vdash |1\rangle:\mathbb{Q}}\,Ax_{|1\rangle}}{\vdash |01\rangle:\mathbb{Q}\otimes\mathbb{Q}}\,\otimes_I}{\vdash \mathsf{H}_{both}\,|01\rangle : S(\mathbb{Q})\otimes S(\mathbb{Q})}\,\Rightarrow_E \tag{7}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(3)}{\vdash \mathsf{H}_1:\mathbb{Q}\otimes\mathbb{Q}\Rightarrow S(\mathbb{Q})\otimes\mathbb{Q}}}{\vdash \mathsf{H}_1:S(\mathbb{Q}\otimes\mathbb{Q}\Rightarrow S(\mathbb{Q})\otimes\mathbb{Q})}\,\preceq \quad \dfrac{\dfrac{\dfrac{(6)}{\vdash U_f:\mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q}}}{\vdash U_f:S(\mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q})}\,\preceq \quad \dfrac{\dfrac{\dfrac{\dfrac{(7)}{\vdash \mathsf{H}_{both}\,|01\rangle : S(\mathbb{Q})\otimes S(\mathbb{Q})}}{\vdash \mathsf{H}_{both}\,|01\rangle : S(S(\mathbb{Q})\otimes S(\mathbb{Q}))}\,\preceq}{\vdash \Uparrow^2_{12}\ \mathbb{Q}\otimes S(\mathbb{Q})\mathsf{H}_{both}\,|01\rangle : S(\mathbb{Q}\otimes S(\mathbb{Q}))}\,\Uparrow_r}{\vdash \Uparrow_2\Uparrow^2_{12}\ \mathsf{H}_{both}\,|01\rangle : S(\mathbb{Q}\otimes\mathbb{Q})}\,\Uparrow_l}{\vdash U_f\ (\Uparrow_2\Uparrow^2_{12}\ \mathsf{H}_{both}\,|01\rangle) : S(\mathbb{Q}\otimes\mathbb{Q})}\,\Rightarrow_{ES}}{\vdash \mathsf{H}_1\ (U_f\ \ \Uparrow_2\Uparrow^2_{12}\ \mathsf{H}_{both}\,|01\rangle) : S(S(\mathbb{Q})\otimes\mathbb{Q})}\,\Rightarrow_{ES}}{\vdash \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q})}\ \mathsf{H}_1\ (U_f\ \ \Uparrow_2\Uparrow^2_{12}\ \mathsf{H}_{both}\,|01\rangle) : S(\mathbb{Q}\otimes\mathbb{Q})}\,\Uparrow_r}{\vdash \mathsf{Deutsch}_f : \mathbb{Q}\otimes S(\mathbb{Q})}\,S_E$$

D Trace and typing of the Teleportation algorithm

The full trace of Teleportation $(\alpha.\,|0\rangle + \beta.\,|1\rangle)$ is given below.
Let $\Uparrow_3 = \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q}))}$, $\Uparrow^{23}_{123} = \Uparrow^{S(\mathbb{Q}\otimes S(\mathbb{Q}\otimes\mathbb{Q}))}_{S(S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q}))}$, $\Uparrow_{23} = \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}_{S(\mathbb{Q}\otimes S(\mathbb{Q}\otimes\mathbb{Q}))}$ and $\Uparrow_1 = \Uparrow^{S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}_{S(S(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q})}$.

$$
\begin{aligned}
&\mathsf{Teleportation}\ (\alpha.\,|0\rangle + \beta.\,|1\rangle)\\
&\quad = \ (\lambda q : S(\mathbb{Q})\ (\mathsf{Bob}\ \Uparrow_3 (\mathsf{Alice}\ (q \otimes \beta_{00})))) \ (\alpha.\,|0\rangle + \beta.\,|1\rangle)\\
&\xrightarrow{(\beta_{\mathsf{n}})}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\mathsf{Alice}\ ((\alpha.\,|0\rangle + \beta.\,|1\rangle) \otimes \beta_{00}))\\
&\xrightarrow{(\beta_{\mathsf{n}})}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ \Uparrow_{23}\Uparrow^{23}_{123} (\alpha.\,|0\rangle + \beta.\,|1\rangle) \otimes \beta_{00})))\\
&\xrightarrow{(\mathsf{dist}^{+}_{\mathsf{r}})}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ \Uparrow_{23} (\Uparrow^{23}_{123} (\alpha.\,|0\rangle \otimes \beta_{00)} + \Uparrow^{23}_{123} (\beta.\,|1\rangle \otimes \beta_{00})))))\\
&\xrightarrow{(\mathsf{dist}^{+}_{\Uparrow})}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ (\Uparrow_{23}\Uparrow^{23}_{123} (\alpha.\,|0\rangle \otimes \beta_{00}) + \Uparrow_{23}\Uparrow^{23}_{123} (\beta.\,|1\rangle \otimes \beta_{00})))))\\
&\xrightarrow{(\mathsf{dist}^{\alpha}_{\mathsf{r}})^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ (\Uparrow_{23} \alpha.\ \Uparrow^{23}_{123} (|0\rangle \otimes \beta_{00}) + \Uparrow_{23} \beta.\ \Uparrow^{23}_{123} (|1\rangle \otimes \beta_{00})))))\\
&\xrightarrow{(\mathsf{neut}^{\Uparrow}_{\mathsf{r}})^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ (\Uparrow_{23} \alpha.(|0\rangle \otimes \beta_{00}) + \Uparrow_{23} \beta.(|1\rangle \otimes \beta_{00})))))\\
&\xrightarrow{(\mathsf{dist}^{\alpha}_{\Uparrow})^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ (\alpha.\ \Uparrow_{23} (|0\rangle \otimes \beta_{00}) + \beta.\ \Uparrow_{23} (|1\rangle \otimes \beta_{00})))))\\
&\xrightarrow{(\mathsf{dist}^{+}_{\mathsf{r}})^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ (\alpha.(\Uparrow_{23} (|0\rangle \otimes (\tfrac{1}{\sqrt{2}}.\,|00\rangle))\\
&\qquad\qquad +(\Uparrow_{23} (|0\rangle \otimes \tfrac{1}{\sqrt{2}}.\,|11\rangle)))\\
&\qquad\qquad +\beta.(\Uparrow_{23} (|1\rangle \otimes (\tfrac{1}{\sqrt{2}}.\,|00\rangle))\\
&\qquad\qquad +(\Uparrow_{23} (|1\rangle \otimes \tfrac{1}{\sqrt{2}}.\,|11\rangle)))))))\\
&\xrightarrow{(\mathsf{dist}^{\alpha}_{\mathsf{r}})^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ (\alpha.(\tfrac{1}{\sqrt{2}}.\ \Uparrow_{23} |000\rangle + \tfrac{1}{\sqrt{2}}.\ \Uparrow_{23} |011\rangle)\\
&\qquad\qquad +\beta.(\tfrac{1}{\sqrt{2}}.\ \Uparrow_{23} |100\rangle + \tfrac{1}{\sqrt{2}}.\ \Uparrow_{23} |111\rangle)))))\\
&\xrightarrow{(\mathsf{neut}^{\Uparrow}_{\mathsf{r}})^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ (\alpha.(\tfrac{1}{\sqrt{2}}.\,|000\rangle + \tfrac{1}{\sqrt{2}}.\,|011\rangle)\\
&\qquad\qquad +\beta.(\tfrac{1}{\sqrt{2}}.\,|100\rangle + \tfrac{1}{\sqrt{2}}.\,|111\rangle)))))\\
&\xrightarrow{(\alpha\mathsf{dist})^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ ((\alpha.\tfrac{1}{\sqrt{2}}.\,|000\rangle + \alpha.\tfrac{1}{\sqrt{2}}.\,|011\rangle)\\
&\qquad\qquad +(\beta.\tfrac{1}{\sqrt{2}}.\,|100\rangle + \beta.\tfrac{1}{\sqrt{2}}.\,|111\rangle)))))\\
&\xrightarrow{(\mathsf{prod})^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ ((\tfrac{\alpha}{\sqrt{2}}.\,|000\rangle + \tfrac{\alpha}{\sqrt{2}}.\,|011\rangle)\\
&\qquad\qquad +(\tfrac{\beta}{\sqrt{2}}.\,|100\rangle + \tfrac{\beta}{\sqrt{2}}.\,|111\rangle)))))\\
&\xrightarrow{(\mathsf{lin}^{+}_{\mathsf{r}})^3}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1((\mathsf{cnot}^3_{12}\ \tfrac{\alpha}{\sqrt{2}}.\,|000\rangle + \mathsf{cnot}^3_{12}\ \tfrac{\alpha}{\sqrt{2}}.\,|011\rangle)\\
&\qquad\qquad +(\mathsf{cnot}^3_{12}\ \tfrac{\beta}{\sqrt{2}}.\,|100\rangle + \mathsf{cnot}^3_{12}\ \tfrac{\beta}{\sqrt{2}}.\,|111\rangle))))\\
&\xrightarrow{(\mathsf{lin}^{\alpha}_{\mathsf{r}})^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}^3_1((\tfrac{\alpha}{\sqrt{2}}.\mathsf{cnot}^3_{12}\,|000\rangle + \tfrac{\alpha}{\sqrt{2}}.\mathsf{cnot}^3_{12}\,|011\rangle)\\
&\qquad\qquad +(\tfrac{\beta}{\sqrt{2}}.\mathsf{cnot}^3_{12}\,|100\rangle + \tfrac{\beta}{\sqrt{2}}.\mathsf{cnot}^3_{12}\,|111\rangle))))
\end{aligned}
$$

$$
\begin{aligned}
\xrightarrow{(\beta_b)^4}_{(1)} \ & \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3((\frac{\alpha}{\sqrt{2}}.((\mathsf{cnot}\ (\mathit{head}\ \ |000\rangle \otimes (\mathit{head}\ \mathit{tail}\ \ |000\rangle))) \otimes (\mathit{tail}\ \mathit{tail}\ \ |000\rangle)) \\
&\qquad\qquad\qquad + \frac{\alpha}{\sqrt{2}}.((\mathsf{cnot}\ (\mathit{head}\ \ |011\rangle \otimes (\mathit{head}\ \mathit{tail}\ \ |011\rangle))) \otimes (\mathit{tail}\ \mathit{tail}\ \ |011\rangle))) \\
&\qquad\qquad\qquad +(\frac{\beta}{\sqrt{2}}.((\mathsf{cnot}\ (\mathit{head}\ \ |100\rangle \otimes (\mathit{head}\ \mathit{tail}\ \ |100\rangle))) \otimes (\mathit{tail}\ \mathit{tail}\ \ |100\rangle)) \\
&\qquad\qquad\qquad + \frac{\beta}{\sqrt{2}}.((\mathsf{cnot}\ (\mathit{head}\ \ |111\rangle \otimes (\mathit{head}\ \mathit{tail}\ \ |111\rangle))) \otimes (\mathit{tail}\ \mathit{tail}\ \ |111\rangle)))))) \\
\xrightarrow{(\mathsf{tail})^{12}}_{(1)} \ & \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3((\frac{\alpha}{\sqrt{2}}.((\mathsf{cnot}\ (\mathit{head}\ \ |000\rangle \otimes (\mathit{head}\ \ |00\rangle))) \otimes (|0\rangle)) \\
&\qquad\qquad\qquad + \frac{\alpha}{\sqrt{2}}.((\mathsf{cnot}\ (\mathit{head}\ \ |011\rangle \otimes (\mathit{head}\ \ |11\rangle))) \otimes (|1\rangle))) \\
&\qquad\qquad\qquad +(\frac{\beta}{\sqrt{2}}.((\mathsf{cnot}\ (\mathit{head}\ \ |100\rangle \otimes (\mathit{head}\ \ |00\rangle))) \otimes (|0\rangle)) \\
&\qquad\qquad\qquad + \frac{\beta}{\sqrt{2}}.((\mathsf{cnot}\ (\mathit{head}\ \ |111\rangle \otimes (\mathit{head}\ \ |11\rangle))) \otimes (|1\rangle)))))) \\
\xrightarrow{(\mathsf{head})^{8}}_{(1)} \ & \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3((\frac{\alpha}{\sqrt{2}}.((\mathsf{cnot}\,|00\rangle) \otimes (|0\rangle)) \\
&\qquad\qquad\qquad + \frac{\alpha}{\sqrt{2}}.((\mathsf{cnot}\,|01\rangle) \otimes (|1\rangle))) \\
&\qquad\qquad\qquad +(\frac{\beta}{\sqrt{2}}.((\mathsf{cnot}\,|10\rangle) \otimes (|0\rangle)) \\
&\qquad\qquad\qquad + \frac{\beta}{\sqrt{2}}.((\mathsf{cnot}\,|11\rangle) \otimes (|1\rangle)))))) \\
\xrightarrow{(\beta_b)^4}_{(1)} \ & \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3(((\frac{\alpha}{\sqrt{2}}.(((\mathit{head}\,|00\rangle) \otimes ((\mathit{head}\,|00\rangle)?(\mathsf{not}(\mathit{tail}\,|00\rangle))\cdot(\mathit{tail}\,|00\rangle))) \otimes |0\rangle) \\
&\qquad\qquad\qquad + \frac{\alpha}{\sqrt{2}}.(((\mathit{head}\,|01\rangle) \otimes ((\mathit{head}\,|01\rangle)?(\mathsf{not}(\mathit{tail}\,|01\rangle))\cdot(\mathit{tail}\,|01\rangle))) \otimes |1\rangle)) \\
&\qquad\qquad\qquad +(\frac{\beta}{\sqrt{2}}.(((\mathit{head}\,|10\rangle) \otimes ((\mathit{head}\,|10\rangle)?(\mathsf{not}(\mathit{tail}\,|10\rangle))\cdot(\mathit{tail}\,|10\rangle))) \otimes |0\rangle) \\
&\qquad\qquad\qquad + \frac{\beta}{\sqrt{2}}.(((\mathit{head}\,|11\rangle) \otimes ((\mathit{head}\,|11\rangle)?(\mathsf{not}(\mathit{tail}\,|11\rangle))\cdot(\mathit{tail}\,|11\rangle))) \otimes |1\rangle)))))) \\
\xrightarrow{(\mathsf{head})^{8}}_{(1)} \ & \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3(((\frac{\alpha}{\sqrt{2}}.((|0\rangle \otimes (|0\rangle?(\mathsf{not}(\mathit{tail}\,|00\rangle))\cdot(\mathit{tail}\,|00\rangle))) \otimes |0\rangle) \\
&\qquad\qquad\qquad + \frac{\alpha}{\sqrt{2}}.((|0\rangle \otimes (|0\rangle?(\mathsf{not}(\mathit{tail}\,|01\rangle))\cdot(\mathit{tail}\,|01\rangle))) \otimes |1\rangle)) \\
&\qquad\qquad\qquad +(\frac{\beta}{\sqrt{2}}.((|1\rangle \otimes (|1\rangle?(\mathsf{not}(\mathit{tail}\,|10\rangle))\cdot(\mathit{tail}\,|10\rangle))) \otimes |0\rangle) \\
&\qquad\qquad\qquad + \frac{\beta}{\sqrt{2}}.((|1\rangle \otimes (|1\rangle?(\mathsf{not}(\mathit{tail}\,|11\rangle))\cdot(\mathit{tail}\,|11\rangle))) \otimes |1\rangle)))))) \\
\xrightarrow{(\mathsf{if}_1)^{2}}_{(1)} \ & \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3(((\frac{\alpha}{\sqrt{2}}.((|0\rangle \otimes (|0\rangle?(\mathsf{not}(\mathit{tail}\,|00\rangle))\cdot(\mathit{tail}\,|00\rangle))) \otimes |0\rangle) \\
&\qquad\qquad\qquad + \frac{\alpha}{\sqrt{2}}.((|0\rangle \otimes (|0\rangle?(\mathsf{not}(\mathit{tail}\,|01\rangle))\cdot(\mathit{tail}\,|01\rangle))) \otimes |1\rangle)) \\
&\qquad\qquad\qquad +(\frac{\beta}{\sqrt{2}}.((|1\rangle \otimes (\mathsf{not}(\mathit{tail}\,|10\rangle))) \otimes |0\rangle) \\
&\qquad\qquad\qquad + \frac{\beta}{\sqrt{2}}.((|1\rangle \otimes (\mathsf{not}(\mathit{tail}\,|11\rangle))) \otimes |1\rangle))))))
\end{aligned}
$$

$$
\begin{aligned}
\xrightarrow{(\mathsf{if}_0)^2}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3(((\frac{\alpha}{\sqrt{2}}.((|0\rangle \otimes (tail\,|00\rangle)) \otimes |0\rangle)\\
+\frac{\alpha}{\sqrt{2}}.((|0\rangle \otimes (tail\,|01\rangle)) \otimes |1\rangle))\\
+(\frac{\beta}{\sqrt{2}}.((|1\rangle \otimes (\mathsf{not}(tail\,|10\rangle))) \otimes |0\rangle)\\
+\frac{\beta}{\sqrt{2}}.((|1\rangle \otimes (\mathsf{not}(tail\,|11\rangle))) \otimes |1\rangle))))))
\end{aligned}
$$

$$
\begin{aligned}
\xrightarrow{(\mathsf{tail})^4}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3(((\frac{\alpha}{\sqrt{2}}.((|0\rangle \otimes |0\rangle) \otimes |0\rangle)\\
+\frac{\alpha}{\sqrt{2}}.((|0\rangle \otimes |1\rangle) \otimes |1\rangle))\\
+(\frac{\beta}{\sqrt{2}}.((|1\rangle \otimes (\mathsf{not}\,|0\rangle)) \otimes |0\rangle)\\
+\frac{\beta}{\sqrt{2}}.((|1\rangle \otimes (\mathsf{not}\,|1\rangle)) \otimes |1\rangle))))))
\end{aligned}
$$

$$
\begin{aligned}
\xrightarrow{(\beta_b)^2}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3(((\frac{\alpha}{\sqrt{2}}.(|0\rangle \otimes |0\rangle \otimes |0\rangle)\\
+\frac{\alpha}{\sqrt{2}}.(|0\rangle \otimes |1\rangle \otimes |1\rangle))\\
+(\frac{\beta}{\sqrt{2}}.(|1\rangle \otimes (|0\rangle ? |0\rangle \cdot |1\rangle) \otimes |0\rangle)\\
+\frac{\beta}{\sqrt{2}}.(|1\rangle \otimes (|1\rangle ? |0\rangle \cdot |1\rangle) \otimes |1\rangle))))))
\end{aligned}
$$

$$
\begin{aligned}
\xrightarrow{(\mathsf{if}_0)}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3(((\frac{\alpha}{\sqrt{2}}.(|0\rangle \otimes |0\rangle \otimes |0\rangle)\\
+\frac{\alpha}{\sqrt{2}}.(|0\rangle \otimes |1\rangle \otimes |1\rangle))\\
+(\frac{\beta}{\sqrt{2}}.(|1\rangle \otimes |1\rangle \otimes |1\rangle)\\
+\frac{\beta}{\sqrt{2}}.(|1\rangle \otimes (|1\rangle ? |0\rangle \cdot |1\rangle) \otimes |1\rangle))))))
\end{aligned}
$$

$$
\xrightarrow{(\mathsf{if}_1)}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 \mathsf{H}_1^3(((\frac{\alpha}{\sqrt{2}}.\,|000\rangle + \frac{\alpha}{\sqrt{2}}.\,|011\rangle) + (\frac{\beta}{\sqrt{2}}.\,|110\rangle + \frac{\beta}{\sqrt{2}}.\,|101\rangle)))))
$$

$$
\xrightarrow{(\mathsf{lin}_r^+)^3}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 ((\mathsf{H}_1^3(\frac{\alpha}{\sqrt{2}}.\,|000\rangle) + \mathsf{H}_1^3(\frac{\alpha}{\sqrt{2}}.\,|011\rangle)) + (\mathsf{H}_1^3(\frac{\beta}{\sqrt{2}}.\,|110\rangle) + \mathsf{H}_1^3(\frac{\beta}{\sqrt{2}}.\,|101\rangle)))))
$$

$$
\xrightarrow{(\mathsf{lin}_r^\alpha)^4}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 ((\frac{\alpha}{\sqrt{2}}.\mathsf{H}_1^3\,|000\rangle + \frac{\alpha}{\sqrt{2}}.\mathsf{H}_1^3\,|011\rangle) + (\frac{\beta}{\sqrt{2}}.\mathsf{H}_1^3\,|110\rangle + \frac{\beta}{\sqrt{2}}.\mathsf{H}_1^3\,|101\rangle))))
$$

$$
\begin{aligned}
\xrightarrow{(\beta_b)^4}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 ((\frac{\alpha}{\sqrt{2}}.((\mathsf{H}(head\,|000\rangle)) \otimes (tail\,|000\rangle)) + \frac{\alpha}{\sqrt{2}}.((\mathsf{H}(head\,|011\rangle)) \otimes (tail\,|011\rangle)))\\
+(\frac{\beta}{\sqrt{2}}.((\mathsf{H}(head\,|110\rangle)) \otimes (tail\,|110\rangle)) + \frac{\beta}{\sqrt{2}}.((\mathsf{H}(head\,|101\rangle)) \otimes (tail\,|101\rangle))))))
\end{aligned}
$$

$$
\begin{aligned}
\xrightarrow{(\mathsf{head})^4}_{(1)}\ \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 ((\frac{\alpha}{\sqrt{2}}.((\mathsf{H}\,|0\rangle) \otimes (tail\,|000\rangle)) + \frac{\alpha}{\sqrt{2}}.((\mathsf{H}\,|0\rangle) \otimes (tail\,|011\rangle)))\\
+(\frac{\beta}{\sqrt{2}}.((\mathsf{H}\,|1\rangle) \otimes (tail\,|110\rangle)) + \frac{\beta}{\sqrt{2}}.((\mathsf{H}\,|1\rangle) \otimes (tail\,|101\rangle))))))
\end{aligned}
$$

$$
\begin{aligned}
&\xrightarrow{(\mathsf{tail})^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 ((\frac{\alpha}{\sqrt{2}}.((\mathsf{H}\,|0\rangle) \otimes |00\rangle) + \frac{\alpha}{\sqrt{2}}.((\mathsf{H}\,|0\rangle) \otimes |11\rangle))\\
&\qquad +(\frac{\beta}{\sqrt{2}}.((\mathsf{H}\,|1\rangle) \otimes |10\rangle) + \frac{\beta}{\sqrt{2}}.((\mathsf{H}\,|1\rangle) \otimes |01\rangle)))))\\
&\xrightarrow{(\beta_b)^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 (\frac{\alpha}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + (|0\rangle ?(-|1\rangle)\cdot|1\rangle)) \otimes |00\rangle)\\
&\qquad +\frac{\alpha}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + (|0\rangle ?(-|1\rangle)\cdot|1\rangle)) \otimes |11\rangle)\\
&\qquad +(\frac{\beta}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + (|1\rangle ?(-|1\rangle)\cdot|1\rangle)) \otimes |10\rangle)\\
&\qquad +\frac{\beta}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + (|1\rangle ?(-|1\rangle)\cdot|1\rangle)) \otimes |01\rangle)))))\\
&\xrightarrow{(\mathsf{if}_0)^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 ((\frac{\alpha}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes |00\rangle)\\
&\qquad +\frac{\alpha}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes |11\rangle))\\
&\qquad +(\frac{\beta}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + (|1\rangle ?(-|1\rangle)\cdot|1\rangle)) \otimes |10\rangle)\\
&\qquad +\frac{\beta}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + (|1\rangle ?(-|1\rangle)\cdot|1\rangle)) \otimes |01\rangle)))))\\
&\xrightarrow{(\mathsf{if}_1)^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(\Uparrow_1 ((\frac{\alpha}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes |00\rangle) + \frac{\alpha}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes |11\rangle))\\
&\qquad +(\frac{\beta}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle) \otimes |10\rangle) + \frac{\beta}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle) \otimes |01\rangle)))))\\
&\xrightarrow{(\mathsf{dist}^+_\Uparrow)^3}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2((\Uparrow_1 \frac{\alpha}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes |00\rangle) + \Uparrow_1 \frac{\alpha}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes |11\rangle))\\
&\qquad +(\Uparrow_1 \frac{\beta}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle) \otimes |10\rangle) + \Uparrow_1 \frac{\beta}{\sqrt{2}}.(\frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle) \otimes |01\rangle))))\\
&\xrightarrow{(\mathsf{dist}^\alpha_\Uparrow)^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2((\frac{\alpha}{\sqrt{2}}.\Uparrow_1 (\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes |00\rangle) + \frac{\alpha}{\sqrt{2}}.\Uparrow_1 (\frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) \otimes |11\rangle))\\
&\qquad +(\frac{\beta}{\sqrt{2}}.\Uparrow_1 (\frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle) \otimes |10\rangle) + \frac{\beta}{\sqrt{2}}.\Uparrow_1 (\frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle) \otimes |01\rangle))))\\
&\xrightarrow{(\mathsf{dist}^\alpha_\Uparrow)^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2((\frac{\alpha}{\sqrt{2}}.\frac{1}{\sqrt{2}}.\Uparrow_1 (|0\rangle + |1\rangle) \otimes |00\rangle + \frac{\alpha}{\sqrt{2}}.\frac{1}{\sqrt{2}}.\Uparrow_1 (|0\rangle + |1\rangle) \otimes |11\rangle)\\
&\qquad +(\frac{\beta}{\sqrt{2}}.\frac{1}{\sqrt{2}}.\Uparrow_1 (|0\rangle - |1\rangle) \otimes |10\rangle + \frac{\beta}{\sqrt{2}}.\frac{1}{\sqrt{2}}.\Uparrow_1 (|0\rangle - |1\rangle) \otimes |01\rangle)))\\
&\xrightarrow{(\mathsf{prod})^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2((\frac{\alpha}{2}.\Uparrow_1 (|0\rangle + |1\rangle) \otimes |00\rangle + \frac{\alpha}{2}.\Uparrow_1 (|0\rangle + |1\rangle) \otimes |11\rangle)\\
&\qquad +(\frac{\beta}{2}.\Uparrow_1 (|0\rangle - |1\rangle) \otimes |10\rangle + \frac{\beta}{2}.\Uparrow_1 (|0\rangle - |1\rangle) \otimes |01\rangle)))\\
&\xrightarrow{(\mathsf{dist}^+_\Uparrow)^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2((\frac{\alpha}{2}.(\Uparrow_1 |000\rangle + \Uparrow_1 |100\rangle) + \frac{\alpha}{2}.(\Uparrow_1 |011\rangle + \Uparrow_1 |111\rangle))\\
&\qquad +(\frac{\beta}{2}.(\Uparrow_1 |010\rangle + \Uparrow_1 (-|110\rangle)) + \frac{\beta}{2}.(\Uparrow_1 |001\rangle + \Uparrow_1 (-|101\rangle)))))\\
&\xrightarrow{(\mathsf{dist}^\alpha_\Uparrow)^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2((\frac{\alpha}{2}.(\Uparrow_1 |000\rangle + \Uparrow_1 |100\rangle) + \frac{\alpha}{2}.(\Uparrow_1 |011\rangle + \Uparrow_1 |111\rangle))\\
&\qquad +(\frac{\beta}{2}.(\Uparrow_1 |010\rangle - \Uparrow_1 |110\rangle) + \frac{\beta}{2}.(\Uparrow_1 |001\rangle - \Uparrow_1 |101\rangle))))
\end{aligned}
$$

$$\xrightarrow{(\mathsf{neut}^{\Uparrow}_{r})^8}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2((\frac{\alpha}{2}.(|000\rangle + |100\rangle) + \frac{\alpha}{2}.(|011\rangle + |111\rangle)) + (\frac{\beta}{2}.(|010\rangle - |110\rangle) + \frac{\beta}{2}.(|001\rangle - |101\rangle))))$$

$$\xrightarrow{(\alpha\mathsf{dist})^4}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(((\frac{\alpha}{2}.|000\rangle + \frac{\alpha}{2}.|100\rangle) + (\frac{\alpha}{2}.|011\rangle + \frac{\alpha}{2}.|111\rangle)) + ((\frac{\beta}{2}.|010\rangle + \frac{\beta}{2}.(-|110\rangle)) + (\frac{\beta}{2}.|001\rangle + \frac{\beta}{2}.(-|101\rangle)))))$$

$$\xrightarrow{(\mathsf{prod})^2}_{(1)} \mathsf{Bob}\ \Uparrow_3 (\pi_2(((\frac{\alpha}{2}.|000\rangle + \frac{\alpha}{2}.|100\rangle) + (\frac{\alpha}{2}.|011\rangle + \frac{\alpha}{2}.|111\rangle)) + ((\frac{\beta}{2}.|010\rangle - \frac{\beta}{2}.|110\rangle) + (\frac{\beta}{2}.|001\rangle - \frac{\beta}{2}.|101\rangle))))$$

$$=_{AC} \mathsf{Bob}\ \Uparrow_3 (\pi_2(((\frac{\alpha}{2}.|000\rangle + \frac{\beta}{2}.|001\rangle) + (\frac{\alpha}{2}.|011\rangle + \frac{\beta}{2}.|010\rangle)) + ((\frac{\alpha}{2}.|100\rangle - \frac{\beta}{2}.|101\rangle) + (\frac{\alpha}{2}.|111\rangle - \frac{\beta}{2}.|110\rangle))))$$

The next rewrite step following rule (proj), may produce one of the following four results probability $\frac{1}{4}$ each:

(00) $\mathsf{Bob}\ \Uparrow_3 |00\rangle \otimes (\alpha.|0\rangle + \beta.|1\rangle)$

(01) $\mathsf{Bob}\ \Uparrow_3 |01\rangle \otimes (\alpha.|1\rangle + \beta.|0\rangle)$

(10) $\mathsf{Bob}\ \Uparrow_3 |01\rangle \otimes (\alpha.|0\rangle - \beta.|1\rangle)$

(11) $\mathsf{Bob}\ \Uparrow_3 |11\rangle \otimes (\alpha.|1\rangle - \beta.|0\rangle)$

So, in general, $\mathsf{Bob}\ \Uparrow_3 |xy\rangle \otimes (\alpha.|z\rangle + [-]\beta.|w\rangle)$. Then,

$$\mathsf{Bob}\ \Uparrow_3 |xy\rangle \otimes (\alpha.|z\rangle + [-]\beta.|w\rangle)$$
$$\xrightarrow{(\mathsf{dist}^+_r)}_{(1)} \mathsf{Bob}(\Uparrow_3 |xy\rangle \otimes \alpha.|z\rangle + \Uparrow_3 |xy\rangle \otimes [-]\beta.|w\rangle)$$
$$\xrightarrow{(\mathsf{dist}^\alpha_r)^2}_{(1)} \mathsf{Bob}\ (\alpha.\Uparrow_3 |xyz\rangle + [-]\beta.\Uparrow_3 |xyw\rangle)$$
$$\xrightarrow{(\mathsf{neut}^{\Uparrow}_{r})^2}_{(1)} \mathsf{Bob}\ (\alpha.|xyz\rangle + [-]\beta.|xyw\rangle)$$
$$\xrightarrow{(\mathsf{lin}^+_r)}_{(1)} (\mathsf{Bob}\ \alpha.|xyz\rangle + \mathsf{Bob}\ [-]\beta.|xyw\rangle)$$
$$\xrightarrow{(\mathsf{lin}^\alpha_r)}_{(1)} (\alpha.\mathsf{Bob}\ |xyz\rangle + [-]\beta.\mathsf{Bob}\ |xyw\rangle)$$
$$\xrightarrow{(\beta_b)^2}_{(1)} (\alpha.\mathsf{Z}^{head|xyz\rangle}(\mathsf{not}^{head\ tail|xyz\rangle}(tail\ tail\ |xyz\rangle)) + [-]\beta.\mathsf{Z}^{head|xyw\rangle}(\mathsf{not}^{head\ tail|xyw\rangle}(tail\ tail\ |xyw\rangle)))$$
$$\xrightarrow{(\mathsf{tail})^6}_{(1)} (\alpha.\mathsf{Z}^{head|xyz\rangle}(\mathsf{not}^{head|yz\rangle}(|z\rangle)) + [-]\beta.\mathsf{Z}^{head|xyw\rangle}(\mathsf{not}^{head|yw\rangle}(|w\rangle)))$$
$$\xrightarrow{(\mathsf{head})^4}_{(1)} (\alpha.\mathsf{Z}^{|x\rangle}(\mathsf{not}^{|y\rangle}\ |z\rangle) + [-]\beta.\mathsf{Z}^{|x\rangle}(\mathsf{not}^{|y\rangle}\ |w\rangle))$$
$$\xrightarrow{(\beta_b)^4}_{(1)} (\alpha.\mathsf{Z}^{|x\rangle}(|y\rangle?\mathsf{not}\,|z\rangle{\cdot}|z\rangle) + [-]\beta.\mathsf{Z}^{|x\rangle}(|y\rangle?\mathsf{not}\,|w\rangle{\cdot}|w\rangle))$$

Cases:

(00) $(\alpha.\mathsf{Z}^{|0\rangle}(|0\rangle?\mathsf{not}\,|0\rangle{\cdot}|0\rangle) + \beta.\mathsf{Z}^{|0\rangle}(|0\rangle?\mathsf{not}\,|1\rangle{\cdot}|1\rangle))$

$$\xrightarrow{(\mathsf{if}_0)^2}_{(1)} (\alpha.\mathsf{Z}^{|0\rangle}\ |0\rangle + \beta.\mathsf{Z}^{|0\rangle}\ |1\rangle)$$
$$\xrightarrow{(\beta_b)^4}_{(1)} (\alpha.|0\rangle?(\mathsf{Z}\,|0\rangle){\cdot}|0\rangle + \beta.|0\rangle?(\mathsf{Z}\,|1\rangle){\cdot}|1\rangle)$$
$$\xrightarrow{(\mathsf{if}_0)^2}_{(1)} (\alpha.|0\rangle + \beta.|1\rangle)$$

$$
\begin{aligned}
\textbf{(01)}\ \ & (\alpha.\mathsf{Z}^{|0\rangle}(|1\rangle?\mathsf{not}\,|1\rangle{\cdot}|1\rangle) + \beta.\mathsf{Z}^{|0\rangle}(|1\rangle?\mathsf{not}\,|0\rangle{\cdot}|0\rangle))\\
& \xrightarrow{(\mathsf{if}_1)^2}_{(1)} (\alpha.\mathsf{Z}^{|0\rangle}(\mathsf{not}\,|1\rangle) + \beta.\mathsf{Z}^{|0\rangle}(\mathsf{not}\,|0\rangle))\\
& \xrightarrow{(\beta_b)^2}_{(1)} (\alpha.\mathsf{Z}^{|0\rangle}(|1\rangle?|0\rangle{\cdot}|1\rangle) + \beta.\mathsf{Z}^{|0\rangle}(|0\rangle?|0\rangle{\cdot}|1\rangle))\\
& \xrightarrow{(\mathsf{if}_1)}_{(1)} (\alpha.\mathsf{Z}^{|0\rangle}\,|0\rangle + \beta.\mathsf{Z}^{|0\rangle}(|0\rangle?|0\rangle{\cdot}|1\rangle))\\
& \xrightarrow{(\mathsf{if}_0)}_{(1)} (\alpha.\mathsf{Z}^{|0\rangle}\,|0\rangle + \beta.\mathsf{Z}^{|0\rangle}\,|1\rangle)\\
& \xrightarrow{(\beta_b)^4}_{(1)} (\alpha.|0\rangle?(\mathsf{Z}\,|0\rangle){\cdot}|0\rangle + \beta.|0\rangle?(\mathsf{Z}\,|1\rangle){\cdot}|1\rangle)\\
& \xrightarrow{(\mathsf{if}_0)^2}_{(1)} (\alpha.\,|0\rangle + \beta.\,|1\rangle)
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(10)}\ \ & (\alpha.\mathsf{Z}^{|1\rangle}(|0\rangle?\mathsf{not}\,|0\rangle{\cdot}|0\rangle) - \beta.\mathsf{Z}^{|1\rangle}(|0\rangle?\mathsf{not}\,|1\rangle{\cdot}|1\rangle))\\
& \xrightarrow{(\mathsf{if}_0)^2}_{(1)} (\alpha.\mathsf{Z}^{|1\rangle}\,|0\rangle - \beta.\mathsf{Z}^{|1\rangle}\,|1\rangle)\\
& \xrightarrow{(\beta_b)^4}_{(1)} (\alpha.|1\rangle?(\mathsf{Z}\,|0\rangle){\cdot}|0\rangle - \beta.|1\rangle?(\mathsf{Z}\,|1\rangle){\cdot}|1\rangle)\\
& \xrightarrow{(\mathsf{if}_1)^2}_{(1)} (\alpha.\mathsf{Z}\,|0\rangle - \beta.\mathsf{Z}\,|1\rangle)\\
& \xrightarrow{(\beta_b)^2}_{(1)} (\alpha.|0\rangle?(-\,|1\rangle){\cdot}|0\rangle - \beta.|1\rangle?(-\,|1\rangle){\cdot}|0\rangle)\\
& \xrightarrow{(\mathsf{if}_0)}_{(1)} (\alpha.\,|0\rangle - \beta.|1\rangle?(-\,|1\rangle){\cdot}|0\rangle)\\
& \xrightarrow{(\mathsf{if}_1)}_{(1)} (\alpha.\,|0\rangle - \beta.(-\,|1\rangle))\\
& \xrightarrow{(\mathsf{prod})}_{(1)} (\alpha.\,|0\rangle + \beta.\,|1\rangle)
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(11)}\ \ & (\alpha.\mathsf{Z}^{|1\rangle}(|1\rangle?\mathsf{not}\,|1\rangle{\cdot}|1\rangle) - \beta.\mathsf{Z}^{|1\rangle}(|1\rangle?\mathsf{not}\,|0\rangle{\cdot}|0\rangle))\\
& \xrightarrow{(\mathsf{if}_1)^2}_{(1)} (\alpha.\mathsf{Z}^{|1\rangle}(\mathsf{not}\,|1\rangle) - \beta.\mathsf{Z}^{|1\rangle}(\mathsf{not}\,|0\rangle))\\
& \xrightarrow{(\beta_b)^2}_{(1)} (\alpha.\mathsf{Z}^{|1\rangle}(|1\rangle?|0\rangle{\cdot}|1\rangle) - \beta.\mathsf{Z}^{|1\rangle}(|0\rangle?|0\rangle{\cdot}|1\rangle))\\
& \xrightarrow{(\mathsf{if}_1)}_{(1)} (\alpha.\mathsf{Z}^{|1\rangle}\,|0\rangle - \beta.\mathsf{Z}^{|1\rangle}(|0\rangle?|0\rangle{\cdot}|1\rangle))\\
& \xrightarrow{(\mathsf{if}_0)}_{(1)} (\alpha.\mathsf{Z}^{|1\rangle}\,|0\rangle - \beta.\mathsf{Z}^{|1\rangle}\,|1\rangle)\\
& \xrightarrow{(\beta_b)^4}_{(1)} (\alpha.|1\rangle?(\mathsf{Z}\,|0\rangle){\cdot}|0\rangle - \beta.|1\rangle?(\mathsf{Z}\,|1\rangle){\cdot}|1\rangle)\\
& \xrightarrow{(\mathsf{if}_1)^2}_{(1)} (\alpha.\mathsf{Z}\,|0\rangle + \beta.\mathsf{Z}\,|1\rangle)\\
& \xrightarrow{(\beta_b)^2}_{(1)} (\alpha.|0\rangle?(-\,|1\rangle){\cdot}|0\rangle - \beta.|1\rangle?(-\,|1\rangle){\cdot}|0\rangle)\\
& \xrightarrow{(\mathsf{if}_0)}_{(1)} (\alpha.\,|0\rangle - \beta.|1\rangle?(-\,|1\rangle){\cdot}|0\rangle)\\
& \xrightarrow{(\mathsf{if}_1)}_{(1)} (\alpha.\,|0\rangle - \beta.(-\,|1\rangle))\\
& \xrightarrow{(\mathsf{prod})}_{(1)} (\alpha.\,|0\rangle + \beta.\,|1\rangle)
\end{aligned}
$$

Hence, in every case, $\mathsf{Teleportation}$ $(\alpha.\,|0\rangle + \beta.\,|1\rangle) \longrightarrow^*_{(1)} (\alpha.\,|0\rangle + \beta.\,|1\rangle)$ as expected.

The typing of $\mathsf{Teleportation}$ is given below:

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\dfrac{}{\vdash ?\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\mathit{If}
\quad
\dfrac{}{y:\mathbb{Q}\vdash y:\mathbb{Q}}\,\mathit{Ax}
}{y:\mathbb{Q}\vdash y?\cdot : \mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_E
}{y:\mathbb{Q}\vdash y?\cdot : S(\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q})}\,\preceq
\quad
\dfrac{\dfrac{}{\vdash |1\rangle : \mathbb{Q}}\,\mathit{Ax}_{|1\rangle}}{\vdash -\,|1\rangle : S(\mathbb{Q})}\,S_I^\alpha
}{y:\mathbb{Q}\vdash y?(-\,|1\rangle)\cdot : S(\mathbb{Q}\Rightarrow\mathbb{Q})}\,\Rightarrow_{ES}
\quad
\dfrac{\dfrac{}{\vdash |0\rangle : \mathbb{Q}}\,\mathit{Ax}_{|0\rangle}}{\vdash |0\rangle : S(\mathbb{Q})}\,\preceq
}{y:\mathbb{Q}\vdash y?(-\,|1\rangle)\cdot|0\rangle : S(\mathbb{Q})}\,\Rightarrow_{ES}
}{\vdash \mathsf{Z} : \mathbb{Q}\Rightarrow S(\mathbb{Q})}\,\Rightarrow_I
\quad
\dfrac{}{x:\mathbb{Q}\vdash x:\mathbb{Q}}\,\mathit{Ax}
}{x:\mathbb{Q}\vdash \mathsf{Z}x : S(\mathbb{Q})}\,\Rightarrow_E
\tag{8}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\mathit{If}\quad\dfrac{}{y:\mathbb{Q}\vdash y:\mathbb{Q}}\,\mathit{Ax}}{y:\mathbb{Q}\vdash y?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\Rightarrow_E}{y:\mathbb{Q}\vdash y?\cdot:S(\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q})}\preceq\quad\dfrac{(8)}{w:\mathbb{Q}\vdash \mathsf{Z}w:S(\mathbb{Q})}}{y:\mathbb{Q},w:\mathbb{Q}\vdash y?\mathsf{Z}w\cdot:S(\mathbb{Q}\Rightarrow\mathbb{Q})}\Rightarrow_{ES}\quad\dfrac{\dfrac{}{v:\mathbb{Q}\vdash v:\mathbb{Q}}\,\mathit{Ax}}{v:\mathbb{Q}\vdash v:S(\mathbb{Q})}\preceq}{y:\mathbb{Q},w:\mathbb{Q},v:\mathbb{Q}\vdash y?\mathsf{Z}w\cdot v:S(\mathbb{Q})}\Rightarrow_{ES}}{y:\mathbb{Q},w:\mathbb{Q}\vdash y?\mathsf{Z}w\cdot w:S(\mathbb{Q})}\,C}{y:\mathbb{Q}\vdash\lambda w:\mathbb{Q}\ y?\mathsf{Z}w\cdot w:\mathbb{Q}\Rightarrow S(\mathbb{Q})}\Rightarrow_I}{\vdash\lambda y:\mathbb{Q}\ \lambda w:\mathbb{Q}\ y?\mathsf{Z}w\cdot w:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow S(\mathbb{Q})}\Rightarrow_I\quad\dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}\,\mathit{Ax}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ x:\mathbb{Q}}\otimes_{E_r}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{Z}^{(\mathit{head}\ x)}:\mathbb{Q}\Rightarrow S(\mathbb{Q})}\Rightarrow_E \tag{9}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash ?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\mathit{If}\quad\dfrac{}{y:\mathbb{Q}\vdash y:\mathbb{Q}}\,\mathit{Ax}}{y:\mathbb{Q}\vdash y?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\Rightarrow_E\quad\dfrac{\dfrac{(4)}{\vdash\mathsf{not}:\mathbb{Q}\Rightarrow\mathbb{Q}}\quad\dfrac{}{z:\mathbb{Q}\vdash z:\mathbb{Q}}\,\mathit{Ax}}{z:\mathbb{Q}\vdash\mathsf{not}\ z:\mathbb{Q}}\Rightarrow_E}{y:\mathbb{Q},z:\mathbb{Q}\vdash y?\mathsf{not}\ z\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}}\Rightarrow_E\quad\dfrac{}{w:\mathbb{Q}\vdash w:\mathbb{Q}}\,\mathit{Ax}}{y:\mathbb{Q},z:\mathbb{Q},w:\mathbb{Q}\vdash y?\mathsf{not}\ z\cdot w:\mathbb{Q}}\Rightarrow_E}{y:\mathbb{Q},z:\mathbb{Q}\vdash y?\mathsf{not}\ z\cdot z:\mathbb{Q}}\,C}{y:\mathbb{Q}\vdash\lambda z:\mathbb{Q}\ y?\mathsf{not}\ z\cdot z:\mathbb{Q}\Rightarrow\mathbb{Q}}\Rightarrow_I}{\vdash\lambda y:\mathbb{Q}\ \lambda z:\mathbb{Q}\ y?\mathsf{not}\ z\cdot z:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\Rightarrow_I\quad\dfrac{\dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}\,\mathit{Ax}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathit{tail}\ x:\mathbb{Q}\otimes\mathbb{Q}}\otimes_{E_l}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathit{head}\ \mathit{tail}\ x:\mathbb{Q}}\otimes_{E_r}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{not}^{(\mathit{head}\ \mathit{tail}\ x)}:\mathbb{Q}\Rightarrow\mathbb{Q}}\Rightarrow_E \tag{10}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(9)}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{Z}^{\mathit{head}\ y)}:\mathbb{Q}\Rightarrow S(\mathbb{Q})}\quad\dfrac{\dfrac{\dfrac{(10)}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{not}^{(\mathit{tail}\ y)}:\mathbb{Q}\Rightarrow\mathbb{Q}}\quad\dfrac{\dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}\,\mathit{Ax}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathit{tail}\ x:\mathbb{Q}\otimes\mathbb{Q}}\otimes_{E_l}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathit{tail}\ \mathit{tail}\ x:\mathbb{Q}}\otimes_{E_l}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{not}^{(\mathit{tail}\ y)}(\mathit{tail}\ \mathit{tail}\ x):\mathbb{Q}}\Rightarrow_E}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{not}^{(\mathit{tail}\ x)}(\mathit{tail}\ \mathit{tail}\ x):\mathbb{Q}}\,C}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{Z}^{\mathit{head}\ y)}(\mathsf{not}^{(\mathit{head}\ \mathit{tail}\ x)}(\mathit{tail}\ \mathit{tail}\ x)):S(\mathbb{Q})}\Rightarrow_E}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{Z}^{\mathit{head}\ x)}(\mathsf{not}^{(\mathit{head}\ \mathit{tail}\ x)}(\mathit{tail}\ \mathit{tail}\ x)):S(\mathbb{Q})}\,C}{\vdash\mathsf{Bob}:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\Rightarrow S(\mathbb{Q})}\Rightarrow_I \tag{11}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(2)}{\vdash\mathsf{H}:\mathbb{Q}\Rightarrow S(\mathbb{Q})}\quad\dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash}\,\mathit{Ax}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathit{head}\ x:\mathbb{Q}}\otimes_{E_r}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathsf{H}(\mathit{head}\ x):S(\mathbb{Q})}\Rightarrow_E\quad\dfrac{\dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash}\,\mathit{Ax}}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash\mathit{tail}\ y:\mathbb{Q}\otimes\mathbb{Q}}\otimes_{E_l}}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathsf{H}(\mathit{head}\ x))\otimes(\mathit{tail}\ y):S(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q}}\otimes_I}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathsf{H}(\mathit{head}\ x))\otimes(\mathit{tail}\ x):S(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q}}\,C}{\vdash\mathsf{H}_1^3:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\Rightarrow S(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q}}\Rightarrow_I \tag{12}$$

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\dfrac{
  \dfrac{
    \dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}}\,Ax
  }{y:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ y:\mathbb{Q}}\,\otimes_{E_r}
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{}{\vdash ?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\mathit{If}
            \quad
            \dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ x:\mathbb{Q}}\,\otimes_{E_r}
          }{x:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{head}\ x)?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_E
          \quad
          \dfrac{
            \dfrac{(4)}{\vdash \mathsf{not}:\mathbb{Q}\Rightarrow\mathbb{Q}}
            \quad
            \dfrac{\dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{tail}\ y:\mathbb{Q}}\,\otimes_{E_l}
          }{y:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathsf{not}(\mathit{tail}\ y):\mathbb{Q}}\,\Rightarrow_E
        }{x:\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{head}\ x)?\mathsf{not}(\mathit{tail}\ y)\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}}\,\Rightarrow_E
      }{x:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{head}\ x)?\mathsf{not}(\mathit{tail}\ x)\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}}\,C
      \quad
      \dfrac{\dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{y:\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{tail}\ y:\mathbb{Q}}\,\otimes_{E_l}
    }{x:\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{head}\ x)?\mathsf{not}(\mathit{tail}\ x)\cdot(\mathit{tail}\ y):\mathbb{Q}}\,\Rightarrow_E
  }{x:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{head}\ x)?\mathsf{not}(\mathit{tail}\ x)\cdot(\mathit{tail}\ x):\mathbb{Q}}\,C
}{
  \dfrac{
    \dfrac{x:\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{heady})\otimes((\mathit{head}\ x)?\mathsf{not}(\mathit{tail}\ x)\cdot(\mathit{tail}\ x)):\mathbb{Q}\otimes\mathbb{Q}}{x:\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{headx})\otimes((\mathit{head}\ x)?\mathsf{not}(\mathit{tail}\ x)\cdot(\mathit{tail}\ x)):\mathbb{Q}\otimes\mathbb{Q}}\,C
  }{\vdash \mathsf{cnot}:\mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q}}\,\Rightarrow_I
}\,\otimes_I
\tag{13}
$$

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\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{(13)}{\vdash \mathsf{cnot}:\mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q}}
      \quad
      \dfrac{
        \dfrac{
          \dfrac{\dfrac{}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ x:\mathbb{Q}}\,\otimes_{E_r}
          \quad
          \dfrac{\dfrac{\dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{tail}\ y:\mathbb{Q}\otimes\mathbb{Q}}\,\otimes_{E_l}}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{head}\ \mathit{tail}\ y:\mathbb{Q}}\,\otimes_{E_r}
        }{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{head}\ x)\otimes(\mathit{head}\ \mathit{tail}\ y):\mathbb{Q}\otimes\mathbb{Q}}\,\otimes_I
      }{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathit{head}\ x)\otimes(\mathit{head}\ \mathit{tail}\ x):\mathbb{Q}\otimes\mathbb{Q}}\,C
    }{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash \mathsf{cnot}((\mathit{head}\ x)\otimes(\mathit{head}\ \mathit{tail}\ x)):\mathbb{Q}\otimes\mathbb{Q}}\,\Rightarrow_E
    \quad
    \dfrac{\dfrac{\dfrac{}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}\,Ax}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{tail}\ y:\mathbb{Q}\otimes\mathbb{Q}}\,\otimes_{E_l}}{y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash \mathit{tail}\ \mathit{tail}\ y:\mathbb{Q}}\,\otimes_{E_l}
  }{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q},y:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathsf{cnot}((\mathit{head}\ x)\otimes(\mathit{head}\ \mathit{tail}\ x)))\otimes(\mathit{tail}\ \mathit{tail}\ x):\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}\,\otimes_I
}{
  \dfrac{x:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\vdash(\mathsf{cnot}((\mathit{head}\ x)\otimes(\mathit{head}\ \mathit{tail}\ x)))\otimes(\mathit{tail}\ \mathit{tail}\ x):\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}{\vdash \mathsf{cnot}^3_{12}:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}\,\Rightarrow_I
}\,C
\tag{14}
$$

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\dfrac{
  \dfrac{\dfrac{(14)}{\vdash \mathsf{cnot}^3_{12}:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}}}{\vdash \mathsf{cnot}^3_{12}:S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}\,\preceq
  \quad
  \dfrac{\dfrac{\dfrac{\dfrac{}{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})}\,Ax}{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash x:S(S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q}))}\,\preceq}{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash_{\Uparrow^{23}_{123}} x:S(\mathbb{Q}\otimes S(\mathbb{Q}\otimes\mathbb{Q}))}\,\Uparrow_r}{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash_{\Uparrow_{23}\Uparrow^{23}_{123}} x:S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}\,\Uparrow_l
}{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash \mathsf{cnot}^3_{12}\ \Uparrow_{23}\Uparrow^{23}_{123}\ x:S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}\,\Rightarrow_{ES}
\tag{15}
$$

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\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{(12)}{\vdash \mathsf{H}^3_1:\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\Rightarrow S(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q}}}{\vdash \mathsf{H}^3_1:S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q}\Rightarrow S(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q})}\,\preceq
      \quad
      \dfrac{(15)}{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash \mathsf{cnot}^3_{12}\ \Uparrow_{23}\Uparrow^{23}_{123}\ x:S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}
    }{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ \Uparrow_{23}\Uparrow^{23}_{123}\ x):S(S(\mathbb{Q})\otimes\mathbb{Q}\otimes\mathbb{Q})}\,\Rightarrow_{ES}
  }{
    \dfrac{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash_{\Uparrow_1} \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ \Uparrow_{23}\Uparrow^{23}_{123}\ x):S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}{x:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\vdash \pi_2(\Uparrow_1 \mathsf{H}^3_1(\mathsf{cnot}^3_{12}\ \Uparrow_{23}\Uparrow^{23}_{123}\ x)):\mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q})}\,S_E
  }\,\Uparrow_r
}{\vdash \mathsf{Alice}:S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q})\Rightarrow\mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q})}\,\Rightarrow_I
\tag{16}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash |0\rangle : \mathbb{Q}}\,Ax_{|0\rangle} \quad \dfrac{}{\vdash |0\rangle : \mathbb{Q}}\,Ax_{|0\rangle}}{\vdash |0\rangle \otimes |0\rangle : \mathbb{Q}\otimes\mathbb{Q}}\,\otimes_I}{\vdash \frac{1}{\sqrt{2}}.\,|0\rangle \otimes |0\rangle : S(\mathbb{Q}\otimes\mathbb{Q})}\,S_I^{\alpha} \quad \dfrac{\dfrac{\dfrac{}{\vdash |1\rangle : \mathbb{Q}}\,Ax_{|1\rangle} \quad \dfrac{}{\vdash |1\rangle : \mathbb{Q}}\,Ax_{|1\rangle}}{\vdash |1\rangle \otimes |1\rangle : \mathbb{Q}\otimes\mathbb{Q}}\,\otimes_I}{\vdash \frac{1}{\sqrt{2}}.\,|1\rangle \otimes |1\rangle : S(\mathbb{Q}\otimes\mathbb{Q})}\,S_I^{\alpha}}{\vdash \beta_{00} : S(S(\mathbb{Q}\otimes\mathbb{Q}))}\,S_I^{+}}{\vdash \beta_{00} : S(\mathbb{Q}\otimes\mathbb{Q})}\,\preceq \tag{17}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(11)}{\vdash \textsf{Bob} : \mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q} \Rightarrow S(\mathbb{Q})}}{\vdash \textsf{Bob} : S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q} \Rightarrow S(\mathbb{Q}))}\,\preceq \quad \dfrac{\dfrac{\dfrac{\dfrac{(16)}{\vdash \textsf{Alice} : S(\mathbb{Q})\otimes S(\mathbb{Q}\otimes\mathbb{Q}) \Rightarrow \mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q})} \quad \dfrac{\dfrac{}{q : S(\mathbb{Q}) \vdash q : S(\mathbb{Q})}\,Ax \quad \dfrac{(17)}{\vdash \beta_{00} : S(\mathbb{Q}\otimes\mathbb{Q})}}{q : S(\mathbb{Q}) \vdash q \otimes \beta_{00} : S(\mathbb{Q}) \otimes S(\mathbb{Q}\otimes\mathbb{Q})}\,\otimes_I}{q : S(\mathbb{Q}) \vdash \textsf{Alice}\ (q\otimes\beta_{00}) : \mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q})}\,\Rightarrow_E}{q : S(\mathbb{Q}) \vdash \textsf{Alice}\ (q\otimes\beta_{00}) : S(\mathbb{Q}\otimes\mathbb{Q}\otimes S(\mathbb{Q}))}\,\preceq}{q : S(\mathbb{Q}) \vdash \Uparrow_1 \textsf{Alice}\ (q\otimes\beta_{00}) : S(\mathbb{Q}\otimes\mathbb{Q}\otimes\mathbb{Q})}\,\Uparrow_l}{q : S(\mathbb{Q}) \vdash \textsf{Bob}\ (\Uparrow_1 \textsf{Alice}\ (q\otimes\beta_{00})) : S(S(\mathbb{Q}))}\,\Rightarrow_{ES}}{q : S(\mathbb{Q}) \vdash \textsf{Bob}\ (\Uparrow_1 \textsf{Alice}\ (q\otimes\beta_{00})) : S(\mathbb{Q})}\,\preceq}{\vdash \mathit{Teleportation} : S(\mathbb{Q}) \Rightarrow S(\mathbb{Q})}\,\Rightarrow_I$$