A vectorial type system

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(DCM 2011)

Journal version in preparation

Algebraic calculi

$\mathbf{t}, \mathbf{r} ::= x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\ \mathbf{r} \mid \mathbf{t} + \mathbf{r} \mid \alpha.\mathbf{t} \mid 0$ $\qquad \alpha \in (S, +, \times)$, a ring.

Two origins:

- Differential λ-calculus [Ehrhard'03]: linearity *à la* Linear Logic

  Removing the differential operator: Algebraic λ-calculus (λ_{alg}) [Vaux'09]

- Quantum computing: superposition of programs

  De-facto linearity: Linear-algebraic λ-calculus (λ_{lin}) [Arrighi,Dowek'08]
  (Non-linear in the sense of LL [Díaz-Caro,Petit'12])

Beta reduction:

$$(\lambda x.\mathbf{t})\ \mathbf{r} \to \mathbf{t}[\mathbf{r}/x]$$

"Algebraic" reductions:

$$\alpha.\mathbf{t} + \beta.\mathbf{t} \to (\alpha + \beta).\mathbf{t},$$
$$\alpha.\beta.\mathbf{t} \to (\alpha \times \beta).\mathbf{t},$$
$$(\mathbf{t})\ (\mathbf{r}_1 + \mathbf{r}_2) \to (\mathbf{t})\ \mathbf{r}_1 + (\mathbf{t})\ \mathbf{r}_2,$$
$$(\mathbf{t}_1 + \mathbf{t}_2)\ \mathbf{r} \to (\mathbf{t}_1)\ \mathbf{r} + (\mathbf{t}_2)\ \mathbf{r},$$
$$\dots$$

(oriented version of the axioms of vectorial spaces)[Arrighi,Dowek'07]

Vectorial space generated by the set of values

$$\mathcal{B} = \{\mathbf{b} : \mathbf{b} \text{ var. or abs. }\}$$

Space of 'results' $::= \mathrm{Span}(\mathcal{B})$

Algebraic calculi

$\mathbf{t}, \mathbf{r} ::= x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\ \mathbf{r} \mid \mathbf{t} + \mathbf{r} \mid \alpha.\mathbf{t} \mid 0$ $\qquad$ $\alpha \in (S, +, \times)$, a ring.

	λ_{alg}	λ_{lin}
Origin	Linear Logic	Quantum computing
Evaluation strategy	Call-by-name	Call-by-value
Algebraic part	Equalities	Rewrite system

CPS simulation [Díaz-Caro,Perdrix,Tasson,Valiron'10]

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Why would it be interesting?

- Several theories using the concept of linear-combination of terms
  quantum, probabilistic, non-deterministic models, ...

Challenge:

A type system capturing the "vectorial" structure of terms

... to check for probability distributions

... or "quantumness" of the term

... or whatever application needing the structure of the vector in normal form

... towards a Curry-Howard approach to defining Fuzzy/Quantum/Probabilistic logics from Fuzzy/Quantum/Probabilistic programming languages.

Encoding quantum computing

Two base vectors: $\begin{array}{l}|0\rangle = \lambda x.\lambda y.x \\ |1\rangle = \lambda x.\lambda y.y\end{array}$

Linear map H s.t. $\begin{array}{l}H|0\rangle = \frac{1}{\sqrt{2}}.|0\rangle + \frac{1}{\sqrt{2}}.|1\rangle \\ H|1\rangle = \frac{1}{\sqrt{2}}.|0\rangle - \frac{1}{\sqrt{2}}.|1\rangle\end{array}$

$$H := \lambda x.\{x[\frac{1}{\sqrt{2}}.|0\rangle + \frac{1}{\sqrt{2}}.|1\rangle][\frac{1}{\sqrt{2}}.|0\rangle - \frac{1}{\sqrt{2}}.|1\rangle]\}$$

$$\begin{array}{ll}[\mathbf{t}] & = \lambda x.\mathbf{t} \\ \{\mathbf{t}\} & = \mathbf{t}(\lambda x.x) \\ \{[\mathbf{t}]\} \to \mathbf{t} & \end{array}$$

The *Scalar* Type System [Arrighi,Díaz-Caro'12]

A polymorphic type system *tracking scalars*:

$$\frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \alpha.T}$$

- Barycentric restrictions
- Characterises the "amount" of terms

The *Additive* Type System [Díaz-Caro,Petit'12]

A polymorphic type system *with sums*:

$$\frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R}$$

- Sums $\sim$ Assoc., comm. pairs
- distributive w.r.t. application

How to combine them?

The *Vectorial* type system

Types:

$$T, R, S := U \mid T + R \mid \alpha.T \mid \mathbb{X}$$
$$U, V, W := \mathrm{X} \mid U \to T \mid \forall \mathrm{X}.U \mid \forall \mathbb{X}.U$$

(U, V, W reflect the values)

Equivalences:

$$\begin{aligned}
1.T &\equiv T \\
\alpha.(\beta.T) &\equiv (\alpha \times \beta).T \\
\alpha.T + \alpha R &\equiv \alpha.(T + R) \\
\alpha.T + \beta.T &\equiv (\alpha + \beta).T \\
T + R &\equiv R + T \\
T + (R + S) &\equiv (T + R) + S
\end{aligned}$$

(reflect the vectorial spaces axioms)

$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash 0 : 0.T}\ 0_I \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T}\ \to_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i.\forall \vec{X}.(U \to T_i) \qquad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{m} \beta_j.U[\vec{A}_j/\vec{X}]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha_i \times \beta_j).T_i[\vec{A}_j/\vec{X}]}\ \to_E$$

$$\frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \alpha.T}\ s_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R}\ +_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad T \equiv R}{\Gamma \vdash \mathbf{t} : R}\ \equiv$$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i.U_i \quad \mathrm{X} \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i.\forall \mathrm{X}.U_i}\ \forall_{\mathrm{I}} \qquad \frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i.\forall \mathrm{X}.U_i}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i.U_i[V/\mathrm{X}]}\ \forall_{\mathrm{E}}$$

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Preliminary version:
`[Arrighi,Díaz-Caro,Valiron'11]`
This revisted version:
`[work-in-progress]`

Subject reduction (in Church-style) ✓
Strong normalisation ✓

The factorisation issue

Or why Curry-style does not work

$$\frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{t} : T'}{\Gamma \vdash \alpha.\mathbf{t} + \beta.\mathbf{t} : \alpha.T + \beta.T'}$$

- However, $\alpha.\mathbf{t} + \beta.\mathbf{t} \to (\alpha + \beta).\mathbf{t}$
- In general $\alpha.T + \beta.T' \neq (\alpha + \beta).T \neq (\alpha + \beta).T'$

(and since we are working in System F, there is no principal types neither)

How 'bad' it is?

i.e. can we stay in Curry?

$$(\alpha + \beta).T \sqsubseteq \alpha.T + \beta.T' \qquad \text{if } \exists\, \mathbf{t} \;/\; \Gamma \vdash \mathbf{t} : T \text{ and } \Gamma \vdash \mathbf{t} : T'$$

(and its contextual closure)

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(and its contextual closure)

Theorem (A weak subject reduction)

If $\Gamma \vdash \boldsymbol{t} : T$ *and* $\boldsymbol{t} \to_R \boldsymbol{r}$*, then*

- *if R is not the factorisation rule:* $\Gamma \vdash \boldsymbol{r} : T$
- *if R is the factorisation rule:* $\exists S \sqsubseteq T \;/\; \Gamma \vdash \boldsymbol{r} : S$

How 'bad' it is?

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$(\alpha+\beta).T \sqsubseteq \alpha.T + \beta.T'$ if $\exists \mathbf{t}$ / $\Gamma \vdash \mathbf{t} : T$ and $\Gamma \vdash \mathbf{t} : T'$
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Theorem (A weak subject reduction)

If $\Gamma \vdash \mathbf{t} : T$ and $\mathbf{t} \to_R \mathbf{r}$, then

- *if R is not the factorisation rule: $\Gamma \vdash \mathbf{r} : T$*
- *if R is the factorisation rule: $\exists S \sqsubseteq T$ / $\Gamma \vdash \mathbf{r} : S$*

How weak?

Let $\mathbf{t} \to \mathbf{r}$,

Subject reduction

$\Gamma \vdash \mathbf{t} : T \Rightarrow \Gamma \vdash \mathbf{r} : T$

Subtyping

$\Gamma \vdash \mathbf{t} : T \Rightarrow \Gamma \vdash \mathbf{r} : S$, but $S \leq T$, so $\Gamma \vdash \mathbf{r} : T$

Our theorem

$\Gamma \vdash \mathbf{t} : T \Rightarrow \Gamma \vdash \mathbf{r} : S$, and $S \sqsubseteq T$

A better solution

Church-style, while more complex, have subject reduction

[Diaz-Caro'11]

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$$H := \lambda x.\underbrace{\{x\underbrace{[\frac{1}{\sqrt{2}}.|0\rangle + \frac{1}{\sqrt{2}}.|1\rangle]}_{I\Rightarrow\boxplus}\underbrace{[\frac{1}{\sqrt{2}}.|0\rangle - \frac{1}{\sqrt{2}}.|1\rangle]}_{I\Rightarrow\boxminus}\}}_{\forall\mathbb{X}.((I\Rightarrow\boxplus)\Rightarrow(I\Rightarrow\boxminus)\Rightarrow(I\Rightarrow\mathbb{X}))\Rightarrow\mathbb{X}}$$

$$[\mathbf{t}] = \lambda x.\mathbf{t}$$
$$\{\mathbf{t}\} = \mathbf{t}(\lambda x.x)$$
$$\{[\mathbf{t}]\} \to \mathbf{t}$$

with
$$\boxplus = \frac{1}{\sqrt{2}}.T + \frac{1}{\sqrt{2}}.F$$
$$\boxminus = \frac{1}{\sqrt{2}}.T - \frac{1}{\sqrt{2}}.F$$

where
$$T = \forall X.\forall Y.X \to Y \to X$$
$$F = \forall X.\forall Y.X \to Y \to Y$$

Most important properties of *Vectorial*

Theorem

If $\vdash \boldsymbol{t} : \sum_i \alpha_i.U_i$ *then* $\boldsymbol{t} \to^* \sum_i \alpha_i.\boldsymbol{b}_i$ *where* $\Gamma \vdash \boldsymbol{b}_i : U_i$

Theorem

If $\boldsymbol{t} \downarrow = \sum_i \alpha_i.\boldsymbol{b}_i$ *then* $\Gamma \vdash \boldsymbol{t} : \sum_i \alpha_i.U_i + 0.T$, *where* $\Gamma \vdash \boldsymbol{b}_i : U_i$

Confluence as a side effect

In the original untyped setting: "confluence by restrictions":

$$Y_{\mathbf{b}} = (\lambda x.(\mathbf{b} + (x)x))\ \lambda x.(\mathbf{b} + (x)x)$$

$$Y_{\mathbf{b}} \to \mathbf{b} + Y_{\mathbf{b}} \to \mathbf{b} + \mathbf{b} + Y_{\mathbf{b}} \to \dots$$

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$$\begin{array}{l} Y_{\mathbf{b}} + (-1).Y_{\mathbf{b}} \ \longrightarrow (1-1).Y_{\mathbf{b}} \ \longrightarrow^{*} 0 \\ \qquad\quad \downarrow \\ \mathbf{b} + Y_{\mathbf{b}} + (-1).Y_{\mathbf{b}} \\ \qquad\quad \downarrow_{*} \\ \qquad\quad \mathbf{b} \end{array}$$

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Solution in the untyped setting:

$\alpha.\mathbf{t} + \beta.\mathbf{t} \to (\alpha + \beta).\mathbf{t}$

only if $\mathbf{t}$ is closed-normal

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In the typed setting: **Strong normalisation solves the problem**

Conclusions

Vectorial type system

- Characterise the vectorial structure of terms
- Church-style leads to subject reduction
- Strong normalisation leads to confluence
- Typable encoding of quantum programs

Future work

- Add restrictions to type ONLY quantum programs
- Towards a quantum Curry-Howard?